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FROBENIUS OVER VALUATION RINGS

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Contents

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Chapter 1

Introduction

Let K a field. An algebraic affine variety is the set of points $x \in K^n$ such that $f_i(x) = 0 \ \forall i \in \{1, \dots, n\}$, where $f_1, \dots, f_n \in R = K[x_1, \dots, x_t]$. Algebraic geometry employs commutative algebra to study properties such as dimension, irreducibility, and smoothness. In this work, we focus on smoothness. A point where an algebraic variety is not smooth, or regular, is called a singularity. In order to identify such points, we study the corresponding quotient ring, R/I where $I = \langle f_1, \dots, f_n \rangle$. In prime characteristic p we employ the *Frobenius morphism*

$$\begin{aligned} F : R &\to R \\ r &\mapsto r^p, \end{aligned}$$

to detect, classify, and measure singularity.

If R is reduced, singularities are studied via a module of p^e-roots R^{1/p^e}. If R is not reduced, there a more general version of this module denoted by $F_*^e R$. A celebrated theorem of Kunz [Kun69] establishes that R is regular if and only if $F_*^e R$ is faithfully flat for every $e \geq 1$ (equivalently for some $e \geq 1$). This theorem opened the door to classify singularities via the structure of $F_*^e R$ as an R-module.

If $F_*^e R$ is a finitely generated R-module, we say R is F-finite. In particular for F-finite local rings, Kunz's Theorem states that R is regular if and only if $F_*^e R$ is a free module. We say that R is strongly F-regular if the growth of the free part of $F_*^e R$ has the same rate as the grow of its rank (see Definition 4.4.1). A ring is F-split if $F_*^e R$ has positive free rank. There are weaker versions of this singularity considering purity of maps instead of free rank.

These are called F-pure regularity and F-purity. These properties have been mostly studied in Noetherian rings. Nonetheless, recently non-Noetherian rings are being considered as well. In particular, Datta and Smith [DS16] studied the behavior of Frobenius map in valuation domains (see Chapter 5). In this thesis we study this topic for both Noetherian and non Notherian valuation domains. In particular, the main goal of this thesis is to give a self-contained exposition of the following result.

Theorem 1.0.1 ([DS16], see Theorem 5.3.4). *Let $(V, \mathfrak{m})$ be a discrete valuation domain with F-finite fraction field. The following are equivalent*

1. *V is F-split;*
2. *V is F-finite;*
3. *V is F-regular.*

To achieve our goal, we focus our attention on four objects: valuations, valuation domains, Frobenius morphism, and their relation. We now present a summary of each chapter of this thesis.

In Chapter 2, we recall some theorems which are used in certain proofs throughout this thesis. References for these theorems are classical books in commutative algebra [Eis95, Mat89, AM69].

Chapter 3 is devoted to study valuation domains and their associated valuations. In particular, we start with valuations and their properties. Later, we define valuations domains and establish the bijection between these two objects. At the end of this chapter, we show the existence of valuations. This material is based on books in integral closure and commutative algebra [HS06, Bou89].

In Chapter 4, we define the Frobenius homomorphism along with the module $F^e_* R$ and its equivalences. In addition, we study several types of singularities according to the behavior of Frobenius map, and state properties and relations among them. We define excellent rings and their relation with Frobenius singularities. For instance, a Noetherian domain is F-finite if and only if it is excellent and its fraction field is F-finite. References for this chapter are notes in Frobenius and methods in prime characteristic [HR76, Smi19].

In Chapter 5 we study valuation domains via Frobenius. By Kunz's Theorem, the Frobenius map is flat for valuation domains. Moreover, these domains are F-pure, and if the valuation domain is excellent and Noetherian, it is also F-split. Hochster and Huneke [HH89a] introduced strongly F-regularity which is only defined for Noetherian rings. Datta and Smith [DS16] introduced a more general concept called F-pure regularity. A valuation domain is F-pure regular if and only if it is Noetherian,. Therefore in the Noetherian case both definitions are equivalent. Finally, we include an example from the paper [DS16] to illustrate that not every valuation domain is F-split (see Example 5.3.7). In particular, there exists a ring that is not F-finite, nor excellent, nor F-regular.

Chapter 2

Background

Troughout this section we give facts used in certain proofs of this work. However, we don't demonstrate each of them since they can be found in many references [Eis95, Mat89, AM69]. First we start by giving theorems which we use in Chapter 3, then theorems used in Chapter 4, finally, theorems used to study Frobenius on valuation rings in Chapter 5. First, we need some definitions.

From algebraic geometry we know that there exists a correspondence between points in $\mathbb{C}$ and maximal ideals in $\mathbb{C}[x]$. This idea can be taken to any ring through the following definition.

Definition 2.0.1. *Let A be a ring. We define the* ***prime spectrum*** *of A as the set*

$$\operatorname{Spec} A = \{P \subseteq A \mid P \text{ prime ideal of } A\}.$$

Remark 2.0.2. Lat A be a ring and I an ideal of A. Then we consider the set

$$V(I) = \{P \in \operatorname{Spec} A \mid I \subseteq P\}.$$

Consider a family of ideals $\{I_\lambda\}_{\lambda \in \Lambda}$. Then

$$\cap_{\lambda \in \Lambda} V(I_\lambda) = V\left(\sum_{\lambda \in \Lambda} I_\lambda\right).$$

Moreover is $I_1, \ldots, I_n$ are ideals, we have that

$$\cup_{i=1}^{n} V(I_i) = V(I_1 \cap \cdots \cap I_n).$$

Therefore, $\operatorname{Spec} A$ is a topological space where $V(I)$ is a closed set $\forall I$ ideal.

Definition 2.0.3. *Let R be a ring. We say R is **normal** if it is reduced and every element of the fraction field of R,* $\text{Frac}(R)$*, that is integral over R is in R.*

If R is a domain, then the integral closure of $\text{Frac}(R)$ is its normalization. This means that we can always get a normal ring out of a domain. We show an example of this.

Example 2.0.4. Consider the ring $R = \mathbb{C}[t^2, t^3]$. We have that $R \cong \frac{\mathbb{C}[x,y]}{(x^3 - y^2)}$. Note that the equation $f(x) = x^3 - y^2$ is de cusp in $\mathbb{R}^2$.

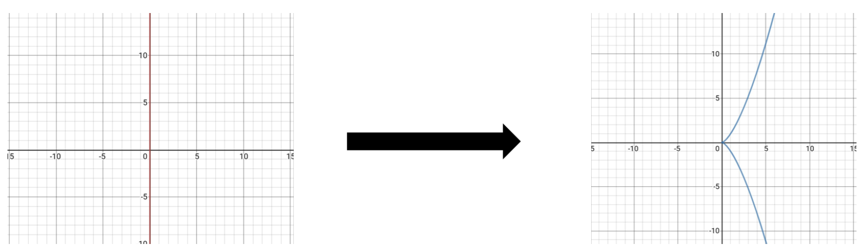
We have that R is not normal since the element t is a integral element over R. Therefore $\overline{R} = \mathbb{C}[t]$. The map

$$R \hookrightarrow \overline{R},$$

induces a map

$$\mathbb{C} \to V(f)$$

sending the i-axis to the cusp.



The normalization of a domain preserves properties of the domain. For example, consider a domain R. If R is a K-algebra finitely generated, with K a field, then

$$\dim R = \dim \overline{R}.$$

To understand this, the following theorem is useful.

Theorem 2.0.5 (Lying over and going up)**.** *Suppose $R \subseteq S$ is an integral extension of rings. Given a prime $P \subseteq R$, there exists a prime $Q \subseteq S$ with $R \cap Q = P$. Moreover, Q may be chosen to contain any given ideal Q_1 satisfying the condition $R \cap Q_1 \subset P$.*

Now we give some definitions in order to talk about Cohen-Macaulay rings.

Definition 2.0.6. *Let R be a ring and P be a prime ideal. The* ***height*** *of P, denoted $\operatorname{ht} P$, is the supremum of lengths of finite strictly ascending chains of prime ideals contained in P.*

Definition 2.0.7. *Let R be a ring, M be a finitely generated module and I be an ideal such that $IM \neq M$. Then the* ***depth*** *of I on M, denoted $\operatorname{depth}(I, M)$, is the length of a maximal M-sequence in I.*

Definition 2.0.8. *Let $(R, \mathfrak{m})$ be a local ring. Then R is said to be* ***Cohen-Macaulay*** *is $\operatorname{depth} \mathfrak{m} = \operatorname{ht} \mathfrak{m}$.*

Furthermore we talk about local rings whose number generators of the maximal ideal is equal to the dimension.

Definition 2.0.9. *Let $(R, \mathfrak{m})$ a Noetherian local ring of dimension d. Then we say that R is* ***regular*** *if $\mathfrak{m}$ can be generated by exactly d elements.*

Remark 2.0.10. We have that a regular Noetherian local ring is an integral domain. In addition the localization at any prime ideal is also regular. If K is a prefect field then any finitely generated K-algebra is regular, in this context we can say R is smooth.

An example of a regular ring is the ring of formal power series. Moreover, Cohen structure theorem states that any regular ring is isomorphic to the ring of formal power of series.

Remark 2.0.11. Every regular ring is Cohen-Macaulay. Any quotient of regular ring with a regular sequence is also Cohen-Macaulay

Serre stated some equivalences for a ring to be normal. In order to do this, he defined the following conditions.

Definition 2.0.12. *Let A be a Noetherian ring. Then we say that A satisfies*

- *the condition (R_i) if A_P is regular for every $P \in \operatorname{Spec}(A)$ with $\operatorname{ht} P \leq i$;*
- *the condition (S_i) if $\operatorname{depth} A_P \geq \min(\operatorname{ht} P, i)$ for every $P \in \operatorname{Spec}(A)$.*

We use a particular version of these equivalences.

Theorem 2.0.13. *A Noetherian ring is normal if and only if it satifies the Serre's condition (R_1) and (S_2).*

Note that if the ring is normal with dimension 1, then it is regular. Moreover, if the dimension is 2, then the ring is Cohen-Macaulay. In addition, Hochster and Huneke [HH89b] proved that if R is (S_2), then $\operatorname{Spec}(R) \setminus V(I)$ is connected for every ideal I such that $\dim(V(I)) \leq \dim(R) - 2$.

The following theorem is a well-known fact about finite ring extensions.

Theorem 2.0.14. *If $R \subseteq S$ are rings such that S is finitely generated as R-module, then $\dim R = \dim S$.*

Krull stated the following theorem. In Chapter 3 we consider a local ring and we apply it to its maximal ideal.

Theorem 2.0.15 (Krull intersection theorem)**.** *Let $I \subseteq R$ be an ideal in a Noetheran ring R. If M is a finitely generated R-module, then there is an element $r \in I$ such that $(1-r)\left(\cap_1^\infty I^j M\right) = 0$. If R is a domain or a local ring, and I is a proper ideal, then*

$$\bigcap_1^\infty I^j = 0.$$

Before giving the next theorem, we need to know what the associated graded ring of an ideal is.

Definition 2.0.16. *Let R be a ring and, I an ideal. The* ***associated graded ring*** *of I is*

$$\operatorname{gr}_I(R) = \oplus_{n\geq 0}\left(I^n/I^{n+1}\right).$$

If R is a Noetherian local ring with maximal ideal $\mathfrak{m}$, the ***fiber cone*** *of I is the ring*

$$\mathcal{F}_I(R) = \frac{R}{\mathfrak{m}} \oplus \frac{I}{\mathfrak{m}I} \oplus \frac{I^2}{\mathfrak{m}I^2} \oplus \cdots$$

The dimension of $\mathcal{F}_I$ is called the ***analytic spread*** *of I.*

Now, we consider when do we have the equality between the dimension of the fiber cone and the dimension of the associated graded ring.

Theorem 2.0.17. *For any ideal I in a local ring $(R, \mathfrak{m})$,*

$$\dim \mathcal{F}_I \leq \dim(\operatorname{gr}_I(R)) = \dim R.$$

Furthermore, if $\mathfrak{m}$ is the maximal ideal in $gr_I(R)$ consisting of all elements of positive degree of $\mathfrak{m}/I$, then

$$\dim(\operatorname{gr}_I(R)) = \operatorname{ht}\mathfrak{m}.$$

Taking some elements in a ring, The Principal Ideal Theorem gives us information about the height of a minimal prime containing these elements.

Theorem 2.0.18 (Principal Ideal Theorem)**.** *Let R be a Noetherian ring. If $x_1, \ldots, x_c \in R$ and P is minimal among primes of R containing $x_1, \ldots, x_c$, then* $\operatorname{codim} P \leq c$.

Krull and Akizuki stated that the integral closure of a Noetherian ring is also Noetherian in the case of finite extensions of one dimensional rings.

Theorem 2.0.19 (Krull-Akizuki Theorem)**.** *If R is a one-dimensional Noetherian domain with quotient field K and L is a finite extension field of K, then any subring S of L that contains R is Noetherian, of dimension at most* 1*, and has only finitely many ideals containing a given nonzero ideal of R. In particular, the integral closure of R in L is Noetherian.*

Although Nakayama's Lemma has many versions, we work with the following one.

Theorem 2.0.20 (Nakayama's Lemma)**.** *Let $(R, \mathfrak{m}, K)$ be a local ring. Given a finitely generated R-module M, note that $M/\mathfrak{m}M$ is a finitely dimensional vector space over K. Then a given set of elements $\{x_1, ..., x_n\} \subseteq M$ is a minimal generating set for M if and only if their classes $\{\bar{x_1}, ..., \bar{x_n}\}$ in $M/\mathfrak{m}M$ are a K-vector space basis.*

Cohen [Coh46] gave an explicit way to describe complete Noetherian local rings.

Theorem 2.0.21 (Cohen Structure Theorem)**.** *Suppose that $(R, \mathfrak{m}, K)$ is a complete local Noetherian ring containing any field. Then R contains a field isomorphic to its residue field and*

$$R \cong K[\![x_1, ..., x_n]\!]/I$$

for some ideal I. The power series variables x_i can be taken to be minimal generators of the maximal ideal. Furthermore, if R is regular then

$$R \cong K[\![x_1, ..., x_n]\!].$$

The Prime Avoidance Theorem provides us a way to take elements outside certain prime ideals.

Theorem 2.0.22 (Prime Avoidance Theorem)**.** *Suppose that $I_1, \ldots, I_n, J$ are ideals of a ring R, and suppose that $J \subset \cup_j I_j$. If R contains an infinite field or if at most two of the I_j are not prime, then J is contained in one of the I_j.*

The following theorem relates a finitely generated module with its localization at maximal ideal. It is also an example of the Local-Global Principle in commutative algebra.

Theorem 2.0.23. *A finitely generated module in a commutative ring is zero if and only if it is zero in the localization at every maximal ideal.*

Next theorem identifies whether a homomorphism is a monomorphism or not based on its localization at maximal ideals.

Theorem 2.0.24. *If $\varphi : M \to N$ is a map of R.modules, then φ is a monomorphism (or ephimorphism or isomorphism) if and only if for every maximal ideal $\mathfrak{m}$ of R the localized map $\varphi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a monomorphism (or ephimorphism or isomorphism).*

Now we relate the property of splitting and the dual of a module.

Theorem 2.0.25. *Consider the R-module homomorphism*

$$\begin{aligned} \sigma : R \to& M \\ 1 \mapsto& m. \end{aligned}$$

Then σ splits if and only if the natural R-module map

$$\begin{aligned} \psi : \mathrm{Hom}\,(M, R) \to& R \\ \phi \mapsto& \phi\,(m) \end{aligned}$$

is surjective.

Whenever we have a finite integral extension of domains, there exists a nonzero homomorphism going backwards. This is stated is the next theorem.

Theorem 2.0.26. *If $B \hookrightarrow R$ is a finite integral extension of domains, then there exists $\phi \in \mathrm{Hom}_B\,(R, B)$ such that $\phi\,(1_R) \neq 0$.*

The next theorem states that a union of subgroups in a directed system is isomorphic to its direct limit, under certain conditions.

Theorem 2.0.27. *Let G be a group and $\{G_i\}_{i \in I}$ a collection of subgroups which form a directed system over a directed set I, that is $i \leq j$ if and only if there exists a map*

$$\varphi_{i,j} : G_i \hookrightarrow G_j.$$

Then $\varinjlim G_i \cong \cup_{i \in I} G_i$.

Chapter 3

Valuation domains

Our goal is to understand the Frobenius morphism in valuation domains. However, before studying these rings, we check on a specific kind of maps called valuations. As one may think, they are strongly related with valuation domains. This relation is established in Section 2.2. In Section 2.3 we give some properties of valuation rings. Finally, the last section states their existence in both Noetherian and non-Noetherian cases.

3.1 Valuations

Valuations are group maps with an additional property. Afterwards, we mention different examples, and how to create a partition out of this set of homomorphisms.

Definition 3.1.1. *Let K be a field and G be a totally ordered Abelian group. A **valuation on** K or a K**-valuation** is a group homomorphism*

$$v : K^* \to G,$$

with the property

$$v\left(x+y\right) \geq \min\left\{v(x), v(y)\right\} \qquad \forall x, y \in K^*, \tag{3.1}$$

*where $K^* := K - \{0\}$. Furthermore, let $L \subseteq K$ be a field extension. We say that v is a **valuation on** K/L, if $v(r) = 0$, $\forall r \in L$.*

Remark 3.1.2. From the properties of group homomorphisms, we get that

1. $v(1) = 0$

2. $v(x^{-1}) = -v(x) \quad \forall x \in K^*$

This definition can be extended to domains, we can consider a domain as follows.

Remark 3.1.3. Let R be a domain with field of fractions K, G be a totally ordered group and

$$v : R \setminus \{0\} \to G$$

be a function such that

1. $v(xy) = v(x) + v(y) \quad \forall x, y \in R,$

2. $v(x + y) \geq \min\{v(x), v(y)\} \quad \forall x, y \in R.$

Then v can be extended uniquely to a valuation on K as follows:

$$\begin{array}{rcc} \tilde{v} : K^* & \longrightarrow & G \\ \dfrac{x}{y} & \longmapsto & v(x) - v(y). \end{array}$$

That's why we also call v a valuation.

Example 3.1.4. Consider $\mathbb{Z}$, p a prime number, and the function

$$\begin{aligned} v_p : \mathbb{Z} &\to \mathbb{Z} \\ m &\mapsto v_p(m) = r, \end{aligned}$$

where r is the biggest power of p that divides m. First note that

$$v_p(m+n) \geq \min\{v(m), v(n)\}.$$

Indeed, let $v_p(m) = r$, $v_p(n) = s$ and $t = \min\{r, s\}$. We have that

$$\begin{aligned} p^t | m \text{ and } p^t | n &\Rightarrow p^t | (n+m) \\ &\Rightarrow t \leq \max\{u \in \mathbb{Z} \mid \quad p^u | (m+n)\} \\ &\Rightarrow t \leq v_p(m+n). \end{aligned}$$

Now we prove that $v_p(mn) = v_p(m) + v_p(n)$. Let $v_p(m) = r$ and $v_p(n) = s$. Then $p^{r+s} | mn$, and $p^{r+s-1} \nmid mn$. Suppose that there exists an element $u > r + s$ such that $p^u | mn$. Since $v_p(m) = r$, we get that $p^{u-r} | n$ which is a contradiction. We conclude that v_p is a $\mathbb{Z}$-valuation.

Lemma 3.1.5. *Let K be a field and v a K-valuation. Then*

$$v(y) = v(-y)$$

for all $y \in K^*$.

Proof. First we show that $v(1) = v(-1)$.

$$v(1) - v(-1) = v(1(-1)^{-1}) = v(-1) \Rightarrow -v(-1) = v(-1) \Rightarrow v(-1) = 0 = v(1)$$

Now, we get that

$$\begin{aligned} v(1) = v(-1) &\Rightarrow v(y) + v(1) = v(y) + v(-1) \\ &\Rightarrow v((y)(1)) = v((y)(-1)) \\ &\Rightarrow v(y) = v(-y). \end{aligned}$$

■

We extend the Property 3.1 for more than just two elements.

Theorem 3.1.6. *Let K be a field, $x_1, ..., x_n \in K$, and v a $K-$valuation. Then,*

1. $v\left(\sum_{i=1}^{n} x_i\right) \geq \min\left\{v\left(x_1\right), ..., v\left(x_n\right)\right\}$
2. *If $v\left(x_i\right)$ are all distinct, then* $v\left(\sum_{i=1}^{n} x_i\right) = \min\left\{v\left(x_1\right), ..., v\left(x_n\right)\right\}.$

Proof.

1. We proceed by induction. Note that the case $n = 2$ is the Property 3.1. Now, suppose it holds for $n-1$ elements. Let $\min\left\{v\left(x_1\right), ..., v\left(x_{n-1}\right)\right\} = v(x_j)$, for some j. Thus,

$$\begin{aligned} v\left(\sum_{i=1}^{n} x_i\right) &\geq \min\left\{v\left(\sum_{i=1}^{n-1} x_i\right), v\left(x_n\right)\right\} \\ &\geq \min\left\{v(x_j), v(x_n)\right\} \\ &= \min\left\{v\left(x_1\right), ..., v\left(x_n\right)\right\}. \end{aligned}$$

2. We proceed by induction on n. We first assume that $n = 2$. Since G is totally ordered, we can assume without loss of generality that $v(x_2) < v(x_1)$. If $v(x_1 + x_2) > v(x_2)$, then

$$v\left(x_2\right) < \min\left\{v\left(x_1\right), v\left(x_2\right)\right\} = v\left(x_2\right),$$

which is a contradiction. Hence $v(x_1 + x_2) = v(x_2)$.

Now suppose the claim holds for $n-1$ elements, and prove it for n. We consider two cases. If $v(\sum_{i=1}^{n-1} x_i) > v(x_n)$, then $v(\sum_{i=1}^{n} x_i) \geq v(x_n)$. We proceed by contradiction. Suppose that $v(\sum_{i=1}^{n} x_i) > v(x_n)$. We have

$$v\left(x_n\right) < \min\left\{v\left(\sum_{i=1}^{n-1} x_i\right), v\left(x_n\right)\right\} = v\left(x_n\right),$$

which is a contradiction.

On the other hand, if $v(x_n) > v\left(\sum_{i=1}^{n-1} x_i\right)$, then

$$v\left(\sum_{i=1}^{n} x_i\right) \geq v\left(\sum_{i=1}^{n-1} x_i\right).$$

We proceed by contradiction. Supposed that $v\left(\sum_{i=1}^{n} x_i\right) > v\left(\sum_{i=1}^{n-1} x_i\right)$. Thus,

$$v\left(\sum_{i=1}^{n-1} x_i\right) < \min\left\{v\left(\sum_{i=1}^{n-1} x_i\right), v\left(x_n\right)\right\} = v\left(\sum_{i=1}^{n-1} x_i\right),$$

which is also a contradiction.

■

We now define a valuation over a polynomial ring, this map will depend on the values assigned to the variables.

Definition 3.1.7. *Let K be a field, and v a valuation on the field of fractions of the polynomial ring $K\left[x_1, ..., x_n\right]$. The valuation is said to be* ***monomial with respect to*** *$x_1, ..., x_n$ if for any polynomial $f = \sum_{j=0}^{m} r_j x^j$, we get*

$$v(f) = \min\left\{v(r_j x^j) \mid \forall j = 0, .., m\right\}.$$

Remark 3.1.8. A monomial valuation is uniquely determined by the values of $x_1, ..., x_n$.

Example 3.1.9. Let K be a field and v a $K(x, y)$-valuation such that $v(x) = \sqrt{2}$ and $v(y) = 1$.

Note that a monomial $x^i y^j$, $i, j \in \mathbb{Z}_+$, has value at least n if and only if $i\sqrt{(2)} + j \geq n$. This is because v is a valuation, and so

$$\begin{aligned} v(x^i y^j) &= v(x^i) + v(y^j) \\ &= iv(x) + jv(y) \\ &= i\sqrt{2} + j. \end{aligned}$$

Then, $i\sqrt{2} + j \geq n$

Now, we give a name to the image of valuations.

Definition 3.1.10. *Let K be a field and v a K-valuation. Then the image $\Gamma_v = v(K^*)$ of v is a totally ordered Abelian group. This is called **the value group** of v.*

Definition 3.1.11. *Let K be a field. We say that valuations*

$$v : K^* \to \Gamma_v \qquad \textit{and} \qquad w : K^* \to \Gamma_w$$

are equivalent if there exists an order preserving group isomorphism

$$\varphi : \Gamma_v \longrightarrow \Gamma_w$$

such that the following diagram commutes

$$\begin{array}{ccc} K & \xrightarrow{v} & \Gamma_v \\ {\scriptstyle w}\downarrow & \swarrow {\scriptstyle \varphi} & \\ \Gamma_w & & \end{array}$$

This definition gives the partition we mention at the begining of this Section.

Example 3.1.12. Let K be a field and $v : K^* \to \mathbb{Z}$. Let $x, y \in K^*$. Suppose $v(x) < v(y)$. Thus, $v(x + y) \geq v(x)$ and $2v(x) < 2v(y)$, and so

$2v(x+y) \geq 2v(x) = \min\{2v(x), 2v(y)\}$. This shows that $2v$ is a valuation and that it preserves order. Consider the function

$$\begin{aligned} \varphi : \mathrm{Im}(v) &\to \mathrm{Im}(2v) \\ v(x) &\mapsto 2v(x), \end{aligned}$$

which is well-defined. Since

$$\begin{aligned} \varphi(v(x)+v(y)) &= 2(v(x)+v(y)) \\ &= 2v(x)+2v(y) \\ &= \varphi(v(x))+\varphi(v(y)), \end{aligned}$$

φ is an homomorphism. Now we prove that it is injective:

$$\begin{aligned} \varphi(v(x)) = \varphi(v(y)) &\Rightarrow 2v(x) = 2v(y) \\ &\Rightarrow v(x) = v(y). \end{aligned}$$

Finally, suppose $y \in \mathrm{Im}(2v)$. Then there exists an element $x \in K^*$ such that $2v(x) = y$. Therefore there exists $v(x) \in \mathrm{Im}(v)$ such that $\varphi(v(x)) = 2v(x) = y$. We conclude that φ is an isomorphism. Then v is equivalent to $2v$.

3.2 Valuation rings

Now we define valuation domains. In addition, we gradually explain the reason why we studied first valuation maps, this is, we prove the relation between valuations and valuation domains.

Definition 3.2.1. *Let K be a field. A **K-valuation domain** is an integral domain V whose field of fractions is K and satisfies the property that, for every non-zero element x in K, either $x \in V$ or $x^{-1} \in V$.*

If it is clear from the context, we omit K.

Example 3.2.2. Let $K = \mathbb{Q}$ and p be a fixed prime. We prove that the set

$$R = \left\{ p^r \frac{m}{n} \in \mathbb{Q} \mid r \geq 0, p \nmid m \text{ and } p \nmid n \right\},$$

is a valuation domain.

Let $\frac{r}{s} \in \mathbb{Q}$ be such that $(r, s) = 1$. Suppose $\frac{r}{s} \notin R$. We show that $\frac{s}{r} \in R$. In the case that $p \nmid r$ and $p \nmid s$, $\frac{s}{r} = \frac{p^0 s}{r} \in R$. Now consider the case when p divides either r or s. Note that if $p \mid r$, then there exist $n, t \in \mathbb{Z}$ such that $r = p^n t$ and $p \nmid t$. Thus $\frac{r}{s} = \frac{p^n t}{s} \in R$ which is a contradiction.

We have that $p \mid s$, and thus there exist $m, t \in \mathbb{Z}$ such that $s = p^m t$ and $p \nmid t$. Hence, $\frac{s}{r} = \frac{p^m t}{r} \in R$.

Proposition 3.2.3. *Let V be a valuation domain. The set of ideals in V is totally ordered by inclusion.*

Proof. Let $I, J \subseteq V$ be ideals and $x \in I \setminus J$. For every element $y \in J \setminus \{0\}$, $\frac{x}{y} \in K$. As V is a valuation domain, $\frac{x}{y} \in V$ or $\frac{y}{x} \in V$. If $\frac{x}{y} \in V$, then $x = \left(\frac{x}{y}\right) y \in J$, which is a contradiction. Hence, $\frac{y}{x} \in V$. This implies that $y = \left(\frac{y}{x}\right) x \in I$. We conclude that $J \subseteq I$. ■

Theorem 3.2.4. *Let V be a valuation domain. Then V has a unique maximal ideal*

$$\mathfrak{m}_v = \left\{x \in V \mid x = 0 \text{ or } x^{-1} \notin V\right\}.$$

Proof. First we show that $\mathfrak{m}_v$ is an ideal. It is a subgroup, because $0 \in \mathfrak{m}_v$, and $x, y \in \mathfrak{m}_v$ implies $x - y \in \mathfrak{m}_v$. Now consider $x \in \mathfrak{m}_v$ and $z \in V$. We prove that $xz \in \mathfrak{m}_v$ by contradiction. If $xz \notin \mathfrak{m}_v$, then $(xz)^{-1} = x^{-1}z^{-1} \in V$. This implies that $(x^{-1}z^{-1})\, z = x^{-1} \in V$, and thus $xz \in \mathfrak{m}_v$.

To show that $\mathfrak{m}_v$ is maximal, suppose that there exists an ideal I such that $\mathfrak{m}_v \subseteq I \subseteq R$ and $\mathfrak{m}_v \neq I$. Let $x \in I \setminus \mathfrak{m}_v$. Then $x^{-1} \in V$, and so $xx^{-1} \in I$. Hence $I = R$. Finally, since the ideals in V are totally ordered, $\mathfrak{m}_v$ is the unique maximal ideal. ■

Remark 3.2.5. Given a valuation we get a valuation domain as follows. Let $v : K^* \to G$ be a valuation, and define the set

$$R_v = \{r \in K^* \mid v(r) \geq 0\} \cup \{0\}.$$

Note that

- $0 \in R_v$;

- if $r, s \in R_v$, then $v(r - s) = \min\{v(r), v(s)\} \geq 0$. Thus $r - s \in R_v$;
- $1 \in R_v$, because $v(1) = 0$;
- if $r, s \in R_v$, then $v(rs) = v(r) + v(s) \geq 0$. Thus $rs \in R_v$.

We deduce that R_v is a subring of K^*. Furthermore, it is an integral domain. Indeed, suppose there exist elements $r, v \in R_v$ such that $rv = 0$, and $r \neq 0$. As $r \in K^*$, we get that $v = 0$. Now consider the set

$$\mathfrak{m}_v = \{r \in K^* \mid v(r) > 0\} \cup \{0\}.$$

It follows that

1. $0 \in \mathfrak{m}_v$;
2. if $r, s \in \mathfrak{m}_v$, then $v(r - s) = \min\{v(r), v(s)\} > 0$, and so $r - s \in V$;
3. if $y \in R_v$, and $r \in \mathfrak{m}_v$, then $v(yr) = v(y) + v(r) > 0$, and so $yr \in \mathfrak{m}_v$.

Thus $\mathfrak{m}_v$ is an ideal of R_v. We prove that it is the unique maximal ideal. Let I be an ideal such that $\mathfrak{m}_v \subseteq I \subseteq R$. Suppose that $\mathfrak{m}_v \subsetneq I$. Then there exists $y \in I$ such that $v(y) = 0$. We get that $y^{-1} \in R_v$. Thus $1 \in I$. We conclude that $\mathfrak{m}_v$ is maximal. Suppose that Q is a maximal ideal of R_v and consider $x \in \mathbb{Q}$. Then,

- if $v(x) = 0$, we get that $x^{-1} \in R_v$. Thus $1 \in Q$, which is a contradiction,
- if $v(x) > 0$, then $x \in \mathfrak{m}_v$. Thus $Q \subseteq \mathfrak{m}_v$, and so $Q = \mathfrak{m}_v$.

Hence, $\mathfrak{m}_v$ is unique. Finally, we prove that R_v is a valuation domain. Let $x \in K$. If $x = 0$, then $x \in R_v$. If x is a non-zero element, then $v(x) < 0$ or $v(x) \geq 0$. When $v(x) < 0$, we get that $x^{-1} \in R_v$. On the other hand, we have that $x \in R_v$, if $v(x) \geq 0$.

Remark 3.2.6. If v and w are equivalent valuations, then there exists an order-preserving isomorphism

$$\varphi : \Gamma_v \to \Gamma_w$$

such that $\varphi(v(x)) = w(x)$, $\forall x \in K^*$. Thus

$$\begin{aligned} x \in R_v &\Leftrightarrow v(x) \geq 0 \\ &\Leftrightarrow \varphi^{-1}(w(x)) \geq 0 \\ &\Leftrightarrow w(x) \geq 0 \\ &\Leftrightarrow x \in R_w \end{aligned}$$

Therefore, $R_v = R_w$.

Definition 3.2.7. *The valuation domain R_v from Remark 3.2.5 is called* ***the valuation ring corresponding to the valuation*** *v and its residue field is denoted by $K(v)$.*

Theorem 3.2.8. *Let V be a valuation domain with field of fractions K, $\Gamma_v = \dfrac{K^*}{V^*}$, where $V^* \subseteq K^*$ is the multiplicative group of units of V. Let*

$$v : K^* \to \Gamma_v$$

be the natural group homomorphism. Then Γ_v is a totally ordered Abelian group, v is a K-valuation, and Γ_v is the value group of v.

Proof. As K^* is an Abelian group under multiplication, Γ_v is also Abelian.

Let $x, y \in K$. We define the relation in Γ_v

$$[x] \leq [y] \Leftrightarrow yx^{-1} \in V.$$

We prove that it is well definied. If $x \sim y$, where $\sim$ is the relation in the quotient, then $xy^{-1}, yx^{-1} \in V^*$. We get that $[x] = [y]$.

Now we show that Γ_v is totally ordered. If $[a], [b] \in \Gamma_v$, then $a, b \in K$. We get that $ab^{-1} \in V$ or $ba^{-1} \in V$. Thus $[a] \leq [b]$ or $[b] \leq [a]$. We conclude that Γ_v is totally ordered.

Now we prove that v is a K-valuation. We know that $v(xy) = v(x) + v(y)$, because v is a group homomorphism. We check that $v(x+y) \geq \min\{v(x), v(y)\}$. Let $x, y \in K^*$. Then, $xy^{-1} \in V$ or $yx^{-1} \in V$. Suppose without loss of generality that $xy^{-1} \in V$, then $(x+y)y^{-1} = xy^{-1} + 1 \in V$. Thus $v(x+y) \geq v(y) \geq \min\{v(x), v(y)\}$.

As v is surjective, we conclude that Γ_v is the value group of v.

∎

Definition 3.2.9. *The valuation map obtained from Theorem 3.2.8 are called* ***associated valuation to the valuation domain*** V*.*

Remark 3.2.10. In Definition 3.2.9, the valuation map is unique up to isomorphism.

Proposition 3.2.11. *If* V *is a* K*-valuation domain and* v *is the valuation obtained from* V*, then the valuation ring of* v *is* V*.*

Proof. We show that $V = R_v$. Let $x \in V$. Then $x \in K$, and so $v(x) \geq 0$ or $v(x) < 0$. In the first case, $x \in R_v$. Note that if $v(x) < 0$, then $v(x) < v(1)$. Thus $x^{-1} \in V$, because of the order we defined in Γ_v, which is a contradiction. We conclude that $x \in R_v$.

Now, let $x \in R_v$. Then $x \in K$ and $v(x) \geq 0$. As V is a valuation domain, thus $x \in V$ or $x^{-1} \in V$. If $x \notin V$, then $x^{-1} \in V$. Since $v(x^{-1}) < 0$, we obtain $v(x^{-1}) < v(1)$. Therefore $x \in V$, which is a contradiction. We conclude that $x \in V$

■

Proposition 3.2.12. *Let* v *be a* K*-valuation and* R_v *the corresponding valuation domain. Then the associated valuation to* R_v *is equivalent to* v*.*

Proof. Let w be the associated valuation to R_v. Consider the map

$$\begin{aligned} \varphi : \frac{K^*}{R_v^*} &\to \Gamma_v \\ [r] &\mapsto v\left(r\right). \end{aligned}$$

Note that it is well defined. If $r, s \in K^*$ are elements such that $r \sim s$, then $rs^{-1} \in R_v^*$. Therefore,

$$0 = v\left(rs^{-1}\right) = v\left(r\right) - v\left(s\right).$$

This is $v\left(r\right) = v\left(s\right)$. Moreover, φ is a group homomorphism.We have that

$$\begin{aligned} \varphi\left([rs]\right) &= v\left(rs\right) \\ &= v\left(r\right) + v\left(s\right) \\ &= \varphi\left([r]\right) + \varphi\left([s]\right). \end{aligned}$$

It is also order preserving. Let $[r], [s] \in \frac{K^*}{R_v^*}$ be such that $[r] < [s]$. Suppose $v\left(s\right) < v\left(r\right)$. Then $v\left(r\right) - v\left(s\right) > 0$, this implies that $v\left(rs^{-1}\right) = 0$. Thus $rs^{-1} \in R_v^*$, and so $[s] \leq [r]$, which is a contradiction. Therefore $v\left(s\right) > v\left(r\right)$.

Now, if $g \in \Gamma_v$. Then there exists $r \in K^*$, such that $x = v(r)$. Therefore $\varphi([r]) = v(r) = x$. This is, φ is surjective. In addition, if $r, s \in K^*$ are elements such that $v(r) = v(s)$, then $v(rs^{-1}) = 0$. This implies that $rs^{-1} \in R_v^*$. Thus, $r \sim s$. We conclude that φ is inyective, and so an order preserving group isomorpshim. Finally, note that

$$v = \varphi \circ w.$$

We conclude that v and w are equivalent.

■

Corollary 3.2.13. *There is a bijection between K-valuation domains and equivalence classes of K-valuations.*

Proof. Let D denote the set of valuation domains and M the set of equivalence classes of K-valuations. Define

$$\begin{aligned} \varphi : M &\to D \\ v &\mapsto R_v \end{aligned}$$

Note that φ is well-defined by Theorem 3.2.11. Consider the map

$$\begin{aligned} \theta : D &\to M \\ V &\mapsto v, \end{aligned}$$

where v is the associated valuation to V. By Theorem 3.2.11, $\varphi \circ \theta = Id_D$. On the other hand by Theorem 3.2.12, $\theta \circ \varphi = Id_M$. ■

Now that we have the relation we were seeking, we use both indistinctively throughtout the rest of this work. In addition, we refer as R_v and Γ_v the valuation domain of the valuation v and its valuation group, respectively.

Proposition 3.2.14. *Let v be a valuation over a field K. Then valuation of a unit in R_v is 0.*

Proof. Let a be a unit in R_v. Then $v(a) \geq 0$. Since $a^{-1} \in R_v$, $v(a^{-1}) = -v(a) \geq 0$, we conclude that $v(a) = 0$. ■

Proposition 3.2.15. *Let v be a valuation over a field K. Then $v(\mathbb{Q}) = 0$.*

Proof. First we prove that if $n \in \mathbb{Z}$, then $v(n) = 0$. We have that for $\frac{1}{n}$,

$$\begin{aligned} v\left(\frac{1}{n}\right) =& v\left(\frac{1}{\underbrace{1+\cdots+1}_{n \text{ times}}}\right) \\ =& v\left(\frac{1}{1}+\cdots+\frac{1}{1}\right) \\ \geq& 0. \end{aligned}$$

Hence, $v(n) = 0$. Now, $v\left(\frac{p}{q}\right) = v(p) + v\left(\frac{1}{q}\right) = v\left(\frac{1}{q}\right)$. Finally, $v\left(\frac{1}{q}\right) = 0$, because $q \in K$. ■

We introduce a kind of groups we often use.

Definition 3.2.16. *Let Γ be a totally ordered Abelian group. We say that Γ is* ***Archimedean****, if for any elements $g, h \in \Gamma$ such that $g > 0$, there exists a positive integer n such that $ng > h$.*

Theorem 3.2.17 (Hölder)**.** *Let Γ be a totally ordered Abelian group that is Archimedean. Then Γ is isomorphic to a subgroup of $\mathbb{R}$.*

Proof. Let $a \in \Gamma$ be fixed positive. Therefore for any $b \in \Gamma$ be positive consider the set

$$S_b = \{r \in \mathbb{Q} | \quad ra \leq b\}$$

Note that S_b is not empty because Γ is Archimedean. In fact, there exists $z \in \mathbb{N}$ such that $zb > a$, and consequently $\frac{1}{z} \in S_b$. Furthermore, as Γ is Archimedean, there exists $n \in \mathbb{N}$ such that $b < na$. Let $r \in S_b$. If $r < n$, then $ra < na$. Therefore, $b < ra < na \leq b$, which is a contradiction. We get that S_b is bounded by n. Thus S_b has a supremum. Define $\varphi : \Gamma \to \mathbb{R}$ by

- $\varphi(0) = 0$,
- $\varphi(b) = \sup S_b$, if $b > 0$,
- $\varphi(-b) = -\varphi(b)$, if $b < 0$.

We show that φ is a homomorphism. Let $b, c \in \Gamma$ be positive elements and

$$A = \{m + n| \ \ m \in S_b, \ n \in S_c\}.$$

As $A \subseteq S_{(b+c)}$, then $\sup A = \sup S_b + \sup S_c \leq \sup S_{(b+c)}$, and so, $\varphi(b) + \varphi(c) \leq \varphi(b + c)$. Now we proceed by contradiction, suppose that $\varphi(b) + \varphi(c) < \varphi(b + c)$. Thus there exists $r, s \in \mathbb{Q}$ such that $\varphi(b) < r$, $\varphi(c) < s$, and $r + s < \varphi(b + c)$. Then $(r + s)a \leq b + c \leq ra + sa$, which is a contradiction. Thus, $\varphi(b) + \varphi(c) = \varphi(b + c)$. If b and c are both negative, then $\varphi(-b) + \varphi(-c) = \varphi(-(b + c))$. Thus,

$$\begin{aligned}\varphi(b) + \varphi(c) &= -(\varphi(-b) + \varphi(-c)) \\ &= -\varphi(-(b + c)) \\ &= \varphi(b + c).\end{aligned}$$

Let $b, c > 0$, $p = \sup S_b$, $q = \sup S_c$, and $r = \sup S_{b-c}$. Since $pa \leq b$ and $qa \leq c$, we get that $(p - q)a \leq b - c$. Thus, $p - q \leq r$. Since $p - q \in S_{b-c}$, $p - q = r$, we have, $\varphi(b) - \varphi(c) = \varphi(b - c)$. Suppose b is a positive element and c is a negative element. Then $-c > 0$. Thus,

$$\begin{aligned}\varphi(b) + \varphi(c) &= \varphi(b) - (-\varphi(c)) \\ &= \varphi(b) - \varphi(-c) \\ &= \varphi(b - (-c)) \\ &= \varphi(b + c).\end{aligned}$$

Now we show that φ preserves inequalities. First, suppose $b < 0$. Then, there exists $m \in \mathbb{N}$ such that $a < mb$. Thus $\dfrac{1}{m} \in S_b$. We get that $0 < \dfrac{1}{m} \leq \varphi(b)$. Now if $c < b$, then $S_c \neq S_b$, so $\sup S_c < \sup S_b$. We conclude that $\varphi(c) < \varphi(b)$. Finally, we prove that φ is injective. Let $x, y \in \Gamma$ be such that $x \neq y$. Then $x < y$ or $y < x$. Without loss of generality suppose $x < y$. Thus, $\varphi(x) < \varphi(y)$ This means that $\varphi(x) \neq \varphi(y)$. Therefore, φ is injective. As φ is surjective over its image, it is isomorphic to a subgroup of $\mathbb{R}$. ■

3.3 Properties of valuation domains.

Theorem 3.3.1. *Every valuation domain is integrally closed.*

Proof. Let $x \in \text{Frac}(V)$ be such that it satifies a polynomial with coefficients in V, i.e.,

$$x^n + r_1 x^{n-1} + ... + r_n = 0.$$

Suppose that $x \notin V$. Then $x^{-1} \in V$. Thus we have that

$$\begin{aligned} x^{-n}\left(x^n + r_1 x^{n-1} + ... + r_n\right) = 0 &\Rightarrow 1 + r_1 x^{-1} + ... + r_n x^{-n} = 0 \\ &\Rightarrow 1 \in x^{-1}V \\ &\Rightarrow x \in V, \end{aligned}$$

which is a contradiction. We conclude that $x \in V$, and so, $V = \text{Frac}(V)$ ∎

Remark 3.3.2. Let V be a K-valuation ring and A be a ring such that $V \subseteq A \subseteq K$. If $x \in K$, then $x \in V \subseteq A$ or $x^{-1} \in V \subseteq A$. Thus A is also a K-valuation ring.

In addition, if we take an element $y \in \mathfrak{m}_A$ such that $y \notin \mathfrak{m}_V$, then $y^{-1} \in V \subseteq A$, which is a contradiction. Therefore, $\mathfrak{m}_A \subseteq \mathfrak{m}_V$, and $\mathfrak{m}_A \in \text{Spec}(V)$. Moreover, consider the localization $V_{\mathfrak{m}_A} \subseteq A$. Let $x \in A \setminus V$. Then $x^{-1} \in V$ and $x^{-1} \notin \mathfrak{m}_A$, so it is a unit in V. We conclude that $V_{\mathfrak{m}_A} = A$.

Proposition 3.3.3. *Let V be a K-valuation ring and let*

$$C = \{A \text{ ring } \mid V \subseteq A \subseteq K\}.$$

Then the map

$$\begin{aligned} \theta : \text{Spec}(V) &\to C \\ P &\mapsto V_P \end{aligned}$$

is a order-reversing bijection. Hence, the set of subrings such that $V \subseteq A \subseteq K$ is totally ordered by inclusion.

Proof. Consider the map

$$\begin{aligned} \psi : C &\to \text{Spec}(V) \\ A &\mapsto \mathfrak{m}_A. \end{aligned}$$

We show that $\theta \circ \psi = Id_C$ and $\psi \circ \theta = Id_{\text{Spec}(V)}$.

Let $P \in \mathrm{Spec}(V)$, and $A = V_P$. By Remark 3.3.2, $\mathfrak{m}_A = P$. We have that

$$\begin{aligned}(\psi \circ \theta)(P) &= \psi(V_P) \\ &= P,\end{aligned}$$

and

$$\begin{aligned}(\theta \circ \psi)(A) &= \psi(\mathfrak{m}_A) \\ &= V_{\mathfrak{m}_A} \\ &= A.\end{aligned}$$

By Remark 3.3.2, the bijection is order-reversing. The last part follows from Proposition 3.2.3. ■

Proposition 3.3.4. *Let V be a K-valuation domain.*

1. *Every finitely generated ideal of V is principal.*
2. *If for some $x, y \in V$, $(x, y)V \neq yV$, then $\forall r \in V$, $(x - ry)V = (x, y)V$.*

Proof.

1. Let I be an ideal of V and $G = \{x_1, x_2, ..., x_n\}$ be a generating set of I. We proceed by induction on n. In the case $n = 1$, we have that $I = \langle x_1 \rangle$. Now, take $n = 2$, and consider $G = \{x, y\}$. Since $x, y \in K$, $xy^{-1} \in V$ or $yx^{-1} \in V$. Thus $(xy^{-1})y = x \in yV$ or $(yx^{-1})x \in xV$, and therefore I is principal. We get that $|G| = 1$.

   Now suppose this holds for $n - 1$. Let $H = \{x_1, x_2, ...x_{n-1}\}$. Then there exists $m \in \{1, 2, ..., n-1\}$ such that $\langle H \rangle = x_m V$. Then $\langle G \rangle = \langle H \rangle + x_n V = x_m V + x_n V$. Applying the case $n = 2$, we get that $\langle G \rangle = x_k V$ for some $k \in \{n, m\}$.

2. Since $(x, y)V \neq yV$, we have that $x \notin yV$ and $x \neq 0$. If $y = 0$, then $(x - ry)V = xV = (x, y)V$, $\forall r \in V$. Now, consider $y \neq 0$. Since $(x, y)V$ is finitely generated, $y \in xV$. Therefore, $(x - ry)V \subseteq xV$. Moreover, there exists $s \in V$ such that $y = sx$. Thus $x - ry = x - rsx = (1 - rs)x$, and $(x - ry)V = (1 - rs)xV$. Note that s is not a unit in V; otherwise $x = s^{-1}y$, which is a contradiction. Thus, $1 - rs$ in a unit in V, this $\forall r \in V$. Hence $x = (1 - rs)(1 - rs)^{-1}x$.

■

Lemma 3.3.5. *Let R be a ring, I be an ideal of R and $V_1, ..., V_n$ be valuation domains that are R-algebras. Assume that for each $j = 1, ..., n$ IV_j is a principal ideal.*

1. *There exist $m \in \mathbb{N}_{>0}$ and $x \in I^m$ such that $\forall i$ $xV_i = I^m V_i$.*

2. *For $i = 1, .., n$ let $\mathfrak{m}_i$ be the maximal ideal V_i. Assume that R contains units $u_1, ..., u_{n-1}$ with the property that modulo each $\mathfrak{m}_i \cap R$ all u_i are disticnt. Then there exists an element $x \in I$ such that $\forall i = 1, ..., n$, $xV_i = IV_i$.*

Proof.

1. We proceed by induction. We first consider the case $n = 1$. Take $m = 1$ and apply the Proposition 3.3.4. Now, suppose our claim holds for $n - 1$. By the case $n = 1$, we get that for all $i \in 1, ..., n$ there exist $m_i \in \mathbb{N}_{>0}$ and an element $x_i \in I^{m_i}$ such that $\forall i \neq j$ $x_i V_j = I^{m_i} V_j$. Define $m = \prod_{i=1}^{n} m_i$, $r_i = \frac{m}{m_i}$, and $x = \sum_{j=1}^{n} x_1^{r_1} ... \widehat{x_j^{r_j}} ... x_n^{r_n}$, where $\widehat{x_j^{r_j}}$ means the element $x_j^{r_j}$ is removed. Note that $x \in I^{m(n-1)}$. Since $I^{m(n-1)}$ is an ideal of R, we get that $I^{m(n-1)} V_i$ is an ideal for all i. Thus by Proposition 3.3.4, $xV_i = I^{m(n-1)} V_i$, $\forall i$.

2. We proceed by contradiction. Consider the case $n = 1$. There are no such units, so we can apply the Proposition 3.3.4. Now, suppose our claim holds for $n - 1$, we show that it holds for n. We may assume that there exist $x, y \in I$ such that for all $1 \leq i < n$ and $1 < j \leq n$, $xV_i = IV_i$ and $yV_j = IV_j$. Note that if $xV_n = IV_n$, we already get what we want, the same with $yV_1 = IV_1$. Thus, suppose that $xV_n \neq IV_n$ and $yV_1 \neq IV_1$. We get that for any unit u in V, by Proposition 3.3.4, $(x - uy) V_n = IV_n$ and $(x - uy) V_1 = IV_1$.

   Now, consider $i = 2, ..., n-1$. Note that if u is a unit and $(x - uy) V_i \neq IV_i$, then $x - uy \in m_i I$. In addition, consider u_k, u_l, with $k \neq l$, such that $x - u_k y$, $x - u_l y \in m_i I$. Then $(x - u_k y) - (x - u_l y) = (u_k - u_l) y \in m_i I$.

   Suppose that $x - u_k y$, $x - u_l y \in m_i I$, for all $k \neq l$. Then $u_k - u_l$ is a unit, and so, $y \in m_i I$. On the other hand, $yV_i = IV_i$, so $y = qy$ for some $q \in m_i$. We get that $y(1-q)=0$, with $y \neq 0$ and $1-q \neq 0$. Hence we have a contradiction, because V_i is a domain. Thus for each i, there exists at most one of the units, u_h such that $x - u_h y \in m_i I$. Recall that we have $n-1$ units and we are considering $n-2$ valuation domains V_i. Then, there is a u_k such that $(x - u_k y) V_i = IV_i$. We conclude that u_k works for $i = 1, ..., n$.

∎

Theorem 3.3.6. *Let $(R, \mathfrak{m})$ be a local domain, K its field of fractions, and $R \neq K$. Then, the following are equivalent:*

1. *R is a Noetherian valuation domain,*
2. *R is a principal ideal domain,*
3. *R is Noetherian and the maximal ideal $\mathfrak{m}$ is principal,*
4. *R is Noetherian and there is no ring properly between R and K,*
5. *R is Noetherian, one-dimensional, and integrally closed,*
6. *$\cap_n \mathfrak{m}^n = 0$ and $\mathfrak{m}$ is principal,*
7. *R is a valuation domain with value group isomophic to a subgroup of $\mathbb{Z}$.*

Proof.

1 ⇒2: Since R is Noetherian, its ideals are finitely generated. Then, by Lemma 3.3.5, the ideals in R are principal.

2 ⇒ 7: First we prove that R is a valuation domain. Let $x \in R$ be such that $\mathfrak{m} = \langle x \rangle$. Note that

$$\mathrm{Frac}(R) = R_x = \left\{ \frac{a}{x^n} \mid a \in R, n \in \mathbb{N} \right\}.$$

Let $f \in \mathrm{Frac}(R)$. Then, $f = \frac{a}{x^n}$ for some $a \in R$, $n \in \mathbb{N}$. Consider $a = rx^t$ for some $r \in R$, $t \in \mathbb{N}$. We have that $f = \dfrac{rx^t}{x^n} = rx^{t-n}$. We have two cases $t > n$ or $n > t$. In the first case, $f \in R$; on the second, $f^{-1} \in R$. We get that R is a valuation domain. Now define the map

$$\begin{aligned} \varphi : \frac{\operatorname{Frac}(R)^*}{R^*} &\to \mathbb{Z} \\ \left[\frac{a}{x^n}\right] &\mapsto n, \end{aligned}$$

Note that φ is well define since $v(r) = 0$ for every $r \in R^*$. In addition, φ is a group homomorphism. Indeed,

$$\begin{aligned} \varphi\left(\left[\frac{a}{x^n}\right]\left[\frac{b}{x^m}\right]\right) &= \varphi\left(\left[\frac{ab}{x^{n+m}}\right]\right) \\ &= n + m \\ &= \varphi\left(\left[\frac{a}{x^n}\right]\right) + \varphi\left(\left[\frac{b}{x^m}\right]\right) \end{aligned}$$

Now we prove that is φ is inyective. Let $\left[\frac{a}{x^n}\right] \in \frac{\operatorname{Frac}(R)^*}{R^*}$ be such that $\varphi\left(\left[\frac{a}{x^n}\right] = 0\right)$. Note that we can take $a \in R \setminus \mathfrak{m}$. Then $n = 0$, this is $\left[\frac{a}{x^n}\right] = a \in R^*$. We conclude that the value group of R is isomorphic to a subgroup of $\mathbb{Z}$.

7 $\Rightarrow$2: Consider the valuation $v : K^* \to K^*/R^* \cong \Gamma \subseteq \mathbb{Z}$. Then there exists $x \in K$ such that $v(x) = 1$. Thus, x is not a unit in R.

Let I be an arbitrary non-zero ideal in R. Since R is the set of all elements with positive valuation and $I \neq \emptyset$, there exists $y \in I$ such that $v(y) = \min\{v(i) | i \in I\}$. Set $n = v(y)$. We have that $v(yx^{-n}) = n - n = 0$. Hence, $yx^{-n} \in R^*$. In addition, $v(zx^{-n}) > 0\ \forall z \in I$, and so, $zx^{-1} \in R$. Note that $z = (zx^{-n})x^n \in x^nR$, $\forall z \in I$. Since $y = rx^n$ for some $r \in R$, $r = yx^{-n}$ in K, we get that $v(r) = 0$. We get that $r \in R^*$ and $x^n \in yR \subseteq I$. Hence, $I = x^nR = yR$. We conclude that R is a principal ideal domain.

7 $\Rightarrow$1: From the previuos implication, we get that R is a principal ideal valuation domain. We conclude R is Noetherian.

We have shown that **1**, **2** and **7** are equivalent.

1 ⇒4: Suppose there exists a ring A such that $R \subseteq A \subsetneq K$. We show that $R = A$. Note that by Remark 3.3.2, A is a valuation domain too. Now let $a \in A$. Then $a \in K$. Thus $a \in R$ or $a^{-1} \in R$. If $a \in R$, then we are done. Suppose that $a^{-1} \in R$ and $a \notin R$. Then $a^{-1} \in A$, and so, a is a unit in A. We get that $a \notin \mathfrak{m}_A$ and $a \in \mathfrak{m}$. Since $\mathfrak{m} = \mathfrak{m}_A \cap R$, $a^{-1} \in \mathfrak{m}$, which is a contradiction. We conclude $a \in R$.

4 ⇒3: Note that R is already Notherian. Now we show that $\dim R = 1$. Let $x \in \mathfrak{m}$. Then we get that $R \subsetneq R_x = K$, where R_x denotes the localization of R in x.
Now consider a non-zero prime ideal Q of R. Then $0 \subsetneq Q \subsetneq R$. As $R_x = K$, we have that $QR_x = K$. This because for each element $y \in Q$, its inverse is in R_x, and so, $QR_x = \langle 1 \rangle = K$.
By Theorem 2.0.13, it suffices to prove that R is normal. By contradiction. Consider $f \in K \setminus R$. Then $R \subsetneq R[f] = K$. Thus, by Theorem 2.0.14 $\dim R = \dim R[f] = 0$, but this is a contradiction. We get that $f \in R$. Hence $\mathfrak{m}$ is principal, and we conclude that $\dim R = 1$.

3 ⇒ 2: Since R is Noetherian, we have that every ideal is finitely generated. Now let $\mathfrak{m} = \langle x \rangle$, with $x \in R$ and let I be any ideal in R. Thus $I = \langle a_1, ...a_n \rangle$. Since $I \subseteq \mathfrak{m}$, we get that

$$a_i = \sum_{j=1}^{k_i} r_{i_j} x^{n_{i_j}} = \left(\sum_{j=1}^{k_i} r_{i_j} x^{n_{i_j} - p_i} \right) x^{p_i},$$

where $p_i = \min \left\{ n_{i_j} \right\}$. Hence $I = \langle x^j \rangle$ with $j = \min \{p_i\}$.

2 ⇒3: Since R is a principal ideal domain, we have that $\mathfrak{m}$ is principal.

Now, we have that **1**, **3**, **4** and **2** are equivalent. Therefore, They are also equivalent to **7**.

1 ⇒5: Since R is Noetherian, it is integrally closed, by Lemma 3.3.1. Hence, since R is a principal domain, it is one dimensional.

5 ⇒6: Applying Krull's Intersection Theorem, we get that $\cap_n \mathfrak{m}^n = 0$. We prove $\mathfrak{m}$ is principal. Note that R is reduced, because it is a domain. Thus it is normal and, by Theorem 2.0.13, it satisfies the Serre's codition (R_1). Now, we know that $R_\mathfrak{m}$ is regular, and so, $\mathfrak{m}R_\mathfrak{m}$ is generated by one element. We conclude that $\mathfrak{m}$ is generated by one element.

6 ⇒2: Let $x \in R$ be such that $\mathfrak{m} = \langle x \rangle$, $I \subsetneq R$ a non-zero ideal, and $y \in I \backslash \{0\}$. As $I \subseteq \mathfrak{m}$, there exist $r_1 \in R$ such that $y = r_1 x$. If $r_1 \in \mathfrak{m}$, we get that $r_1 = r_2 x$, $r_2 \in R$, and so, $y = r_2 x^2$. Inductively there exists $n \in \mathbb{Z}$ such that $r_n \notin \mathfrak{m}$ and $y = r_n x^n$; otherwise, $y \in \cap \mathfrak{m}^n$ but this is not possible.

Thus, r_n is a unit, and so, $\langle y \rangle = \langle x^n \rangle$. Since this holds for every nonzero element in I, we can consider the least such integer k, such that $\langle z \rangle = \langle x^k \rangle$ for each element $z \in I$. We conclude that $I = \langle x^k \rangle$.

Finally, we get that **1**, **5**, **6** and **2** are equivalent. We conclude the equivalence among the statements. ■

Proposition 3.3.7. *A K-valuation domain V is Noetherian if and only if $\Gamma \cong \mathbb{Z}$ or $\Gamma \cong \{0\}$.*

Proof. Let Γ be the value group obtained from V. By Theorem 3.3.6, we know that $V \neq K$ is Noetherian if and only if $\Gamma \cong \mathbb{Z}$.

Now if $V = K$, then $\frac{K^*}{V^*} = \frac{K^*}{K^*} = 0$, so $\Gamma = \{0\}$. On the other hand, if $\Gamma \cong \{0\}$, then $\frac{K^*}{V^*} = V^*$. Thus $\forall x \in K^*$, $x \in V^*$. We get that $V = K$. Therefore, $V = K$ if and only if $\Gamma \cong \{0\}$. ■

Definition 3.3.8. *A valuation and its corresponding valuation domain are said to be **(generalized) discrete** if its value group is isomorphic to $\mathbb{Z}^n$ with the lexicographic order.*

Recall that our goal is to prove the main theorem over a discrete valuation domains under certain conditions.

Example 3.3.9. Some examples of discrete valuation domains are

- $K[\![x]\!]$ with K a field,
- $\mathcal{Q}_p$ with p prime.

Now we relate Noetherian and discrete valuation domains.

Theorem 3.3.10. *Let V be a valuation domain that is not a field. Then V is Noetherian if and only if it is a discrete valuation domain of dimension one.*

Proof. First, suppose V is Noetherian. Then, by Theorem 3.3.6, V has dimension one and its valuation group is isomorphic to a subgroup of $\mathbb{Z}$.

Now, suppose V is a discrete valuation domain of dimension one. We show that V is a principal ideal domain. Let I be an ideal in V. There exists an element $y \in I$ such that

$$v(y) = \min\{v(i) \mid i \in I\},$$

where v is the associated valuation to V. Note that $(y) \subseteq I$. Now, let $z \in I$. We have that $v(z) \geq v(y)$. Then $v(zy^{-1}) \geq 0$, and so, $zy^{-1} \in V$. As $z = (zy^{-1})\,y \in (y)$, we conclude that $I \subseteq (y)$. ∎

Now we want to understand valuation domians over field extensions, for that we have the following lemma.

Lemma 3.3.11. *The value group of a one dimensional valuation ring V is isomorphic to a subgroup of $\mathbb{R}$.*

Proof. Let Γ be the value group we obtained from V, we need to prove it is Archimedean.

We proceed by contradiction. Let $g, h \in \Gamma$ such that $g > 0$. Consider $x, y \in V$ be such that $v(x) = g$ and $v(y) = h$. Suppose that $ng < h$, $\forall n \in \mathbb{N}$. Thus $\langle y \rangle$ is a non-zero ideal. Note that if $x \in \langle y^m \rangle$, for some $m \in \mathbb{N}$, then $x = ry^m$ with $r \in V$. Thus $g = v(x) = v(y) = v(y^m) = mh$, which is a contradiction. Hence, $x \notin \sqrt{\langle y \rangle}$.

Since $\sqrt{\langle y \rangle} = \mathfrak{m}_V$, we deduce that x is a unit. Hence $v(x) = 0$, which is a contradiction. We conclude that there exists $m \in \mathbb{N}$ such that $mg > h$. By Theorem 3.2.17, Γ is isomorphic to a subgroup of $\mathbb{R}$. ∎

Proposition 3.3.12. *Let V be a K-valuation domain and F be a subfield of K. Then*

1. *the intersection $V \cap F$ is a F-valuation domain;*
2. *if V is Noetherian, then so is $V \cap F$;*
3. *if $F \subseteq K$ is an algebraic extension, then $\Gamma_V \bigotimes_{\mathbb{Z}} \mathbb{Q} = \Gamma_{V \cap F} \bigotimes_{\mathbb{Z}} \mathbb{Q}$.*

Proof.

1. Let $x \in F$. Then $x \in K$. Thus, $x \in V$ or $x^{-1} \in V$. If $x \in V$, we are done. Suppose $x \notin V$. Then $x^{-1} \in V$. Since F is a field, $x^{-1} \in F$, and $x^{-1} \in V \cap F$. We conclude $V \cap F$ is an F-valuation domain. Note that the corresponding valuation is the restriction of the valuation v in F.

2. First we prove $\Gamma_{V\cap F} \subseteq \Gamma_V$. Consider $g \in \Gamma_{V\cap F}$. Then there exists $x \in V \cap F$ such that $v|_F(x) = g$. Since $x \in V$, we have that $v(x) = g \in \Gamma_V$. Now, by Theorem 3.3.6, Γ_V is isomorphic to a subgroup of $\mathbb{Z}$. Since $\Gamma_{V\cap F}$ is a subgroup of Γ_V, we deduce that it is isomorphic to a subgroup of $\mathbb{Z}$. Hence $V \cap F$ is Noetherian.

3. Let $F \subseteq K$ be an algebraic extension and $x \in K$. Then there exists a polynomial such that

$$x^n + a_1x^{n-1} + \cdots + a_n = 0$$

with $a_i \in F$. Note that

$$v\left(a_ix^{n-i}\right) = v\left(a_i\right) + v\left(x^{n-i}\right) = \left(n-i\right)v\left(x\right),$$

because $v\left(a_n\right) = 1$. Therefore, $v\left(a_ix^{n-i}\right) \neq v\left(a_jx^{n-j}\right)$ for every $i, j \in \{1, \ldots, n\}$, with $i \neq j$. Hence we have that

$$\begin{aligned}
&a_ix^{n-i} = -x^n - \cdots - \widehat{a_ix^{n-i}} - \cdots - a_n \\
\Rightarrow &v\left(a_ix^{n-i}\right) = v\left(-x^n - \cdots - \widehat{a_ix^{n-i}} - \cdots - a_n\right) \geq \min\left\{v\left(a_jx^{n-j}\right) | j \neq i\right\} \\
\Rightarrow &v\left(a_ix^{n-i}\right) = v\left(a_jx^{n-j}\right) \text{ for some } j \\
\Rightarrow &\left(i-j\right)v(x) = v\left(a_i\right) - v\left(a_j\right) \in \Gamma_{V\cap F} \\
\Rightarrow &v(x) \otimes 1 = (i-j)v(x) \otimes \frac{1}{i-j}.
\end{aligned}$$

We conclude that $\Gamma_V \otimes_{\mathbb{Z}} \mathbb{Q} = \Gamma_{V\cap F} \otimes_{\mathbb{Z}} \mathbb{Q}$

■

Theorem 3.3.13. *Let V be a valuation ring with maximal ideal $\mathfrak{m}$ and W be the $\mathfrak{m}$-adic completion of V. Then W is a valuation ring.*

Proof. First we need to prove W is a domain. Consider $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ two Cauchy sequences of elements in V whose product converges to zero in W. Then for $N > 0$, there exists $M \in \mathbb{Z}$ such that $\forall n \geq M$, $a_n b_n \in \mathfrak{m}^{2N}$. We have

$$a_n b_n = \sum_{i=0}^{l} c_i,$$

with $c_i \in \mathfrak{m}^{2N}$. Let $I^{2N} = \langle c_i \mid i = 1, \dots t\rangle \subseteq \mathfrak{m}$. Then $a_n b_n \in I$. By Proposition 3.3.4, there exists an element c_n such that $I^{2N} = \langle c_n\rangle^{2N}$. Note that for each n, we have that $a_n \in \langle c_n\rangle^N$ or $b_n \in \langle c_n\rangle^N$.

As we are working with Cauchy sequences, there exists an integer T such that $a_n - a_{n+1} \in \mathfrak{m}^T$ and $b_n - b_{n+1} \in \mathfrak{m}^T$. Consider an integer $n_0 \geq \max\{M, T\}$, and without loss of generality suppose that $a_{n0} \in C_{n0}^N \subseteq \mathfrak{m}^N$. Then for every $n \geq n_0$, $a_n \in \mathfrak{m}^N$. Then, the sequence $\{a_n\}_{n\in\mathbb{N}}$ is zero. Hence W is a domain.

Now, let $x \in \mathrm{Frac}(W)$. Then

$$x = \frac{\{a_n\}_{n\in\mathbb{N}}}{\{b_n\}_{n\in\mathbb{N}}}$$

where $\{a_n\}_n \in \mathbb{N}$ and $\{b_n\}_n \in \mathbb{N}$ are two Cauchy sequences in V.

As the value group is totally order we have that $v(a_n) \leq v(b_n)$ or $v(b_n) \leq v(a_n)$ for each $n \in \mathbb{N}$. ■

From this point, we give some interesting properties of valuations. For further material refer to Nicolas Bourbaki's book in Commutative Algebra [Bou89].

Definition 3.3.14. *Let v be a K-valuation, $a \in K$, and $g \in \Gamma_v$. Define the set*

$$B_g(a) = \{b \in K \mid v(a-b) > g\}.$$

This set is the base of ***the topology defined by*** *v.*

Definition 3.3.15. *Consider an extension of fields $K \subseteq L$. A L-valuation w is called an* ***extension*** *of a K-valuation v if $w|_K = v$. Likewise, we say that R_w* ***dominates*** *R_v, if $R_v = R_w \cap K$ with $\mathfrak{m}_w \cap R_v = \mathfrak{m}_v$.*

Remark 3.3.16. Note that we have the following maps

$$\frac{R_v}{\mathfrak{m}_v} \hookrightarrow \frac{R_w}{\mathfrak{m}_w}.$$

$$\Gamma_v \hookrightarrow \Gamma_w$$

Definition 3.3.17. *The degree of the field extension in Remark 3.3.16 is called* ***residue degree of*** w ***over*** v*, and it is denoted* $f(w/v)$*.*

The ***ramification index of*** w ***over*** v*, denoted* $e(w/v)$*, is the index of* Γ_v *in* Γ_w *.*

Remark 3.3.18. If $K \hookrightarrow L$ is a finite extension, then $f(w/v)$ and $e(w/v)$ are both finite.

Definition 3.3.19. *Let* V *and* W *be two* K*-valuations. We say* V *and* W *are* ***independent****, if* K *is the ring generated by* V *and* W*; otherwise, they are* ***dependent****. Similarly, two valuations* v *and* w *are* ***independent (dependent)*** *if their rings are independent (dependent).*

Remark 3.3.20. Dependence is na equivalence relation.

Proposition 3.3.21. *Let* v*,*w *be* K*-valuations. Then* v *and* w *are dependent if and only if they define the same topology.*

Proof. First suppose they define the same topology. Then consider the preserving order map as the identity.

For the converse, suppose that v and w are dependent. Therefore, there exists a order preserving group homomorphism ϕ. Let $\gamma_1 \in \Gamma_v$, $\gamma_2 \in \Gamma_w$, and $a \in K$. Then we have the sets $B_{\gamma_1}(a)$ and $B_{\gamma_1}(a)$ as in Definition 3.3.14.

Take $b \in B_{\gamma_1}(a)$. Then $v(a-b) > \gamma_1$. We have that

$$w(a-b) > \phi(v(a-b)) > \phi(\gamma_1).$$

We conclude they define the same topology. ■

Lemma 3.3.22. *Let* $v_1, \ldots, v_n$ *with* $n \geq 2$ *be pairwise dependent* K*-valuations. Then the corresponding rings* $V_1, \ldots, V_n$ *generate a subring of* K *distinct from* K*.*

Proof. We proceed by induction. The case $n = 2$, follows from the definition. Suppose our claim holds for $n-1$ valuations. Then, there exists a subring $A \subsetneq K$ such that $V_i \subsetneq A$ for $i = 1, \ldots, n-1$. On the other hand, there exists a subring $B \subsetneq K$ such that $V_{n-1} \subsetneq B$ and $V_n \subsetneq B$. By Proposition 3.3.3, A and B are comparable with the inclusion. The greater is the subring of K we are looking for. ■

3.4 Existence of valuation rings

Lemma 3.4.1. *Let R be a domain with field of fractions K. Let $\mathfrak{m}$ be a prime ideal of R. Then for all $x \in K^*$, either $\mathfrak{m}R[x] \neq R[x]$ or $\mathfrak{m}R[x^{-1}] \neq R[x^{-1}]$.*

Proof. Localizing at $\mathfrak{m}$ we assume that $\mathfrak{m}R[x^{-1}] = R[x^{-1}]$, we show that $\mathfrak{m}R[x] \neq R[x]$. We have that

$$\begin{aligned} 1 = a_0 + a_1x^{-1} + \cdots + a_nx^{-n} &\Rightarrow x^n = a_0x^n + a_1x^{n-1} + \cdots + a_n \\ &\Rightarrow (1 - a_0)\, x^n = a_1x^{n-1} + \cdots + a_n \\ &\Rightarrow 0 = -(1 - a_0)\, x^n + a_1x^{n-1} + \cdots + a_n, \end{aligned}$$

for some $a_i \in \mathfrak{m}$. Since $a_0 \in \mathfrak{m}$, $(1 - a_0)$ is a unit in R, we get that x is integral over R. Hence $R[x]$ is an integral extension of R. By the Lying Over Theorem, there exists a prime ideal $\mathfrak{n} \subsetneq R[x]$, such that $\mathfrak{n} \cap R = \mathfrak{m}$. We conclude that $\mathfrak{m}R[x] \neq R[x]$. ■

Theorem 3.4.2. *Let R be an integral domain, and let $\mathfrak{m}$ be a non-zero prime ideal in R. Then, there exists a valuation domain V between R and the field of fractions of R, such that $\mathfrak{m}_V \cap R = \mathfrak{m}$, where $\mathfrak{m}_V$ is the maximal ideal of V.*

Proof. Localizing at $\mathfrak{m}$ we may assume that R is local, and K is its quotient field. Consider the set

$$\Pi = \{(A, \mathfrak{m}_A) \text{ local rings} | R \subseteq A, \mathfrak{m}A \subseteq \mathfrak{m}_A, \text{ and } A \subseteq K\}.$$

Note that $(R, \mathfrak{m}) \in \Pi$. Now, we consider the following partial order:

$$(A, \mathfrak{m}_A) \leq (B, \mathfrak{m}_B) \Leftrightarrow A \subseteq B \text{ and } \mathfrak{m}_AB \subseteq \mathfrak{m}_B.$$

Take $(A_0, \mathfrak{m}_{A_0}) \leq (A_1, \mathfrak{m}_{A_1}) \leq \dots$ an ascending chain and note that the element $\cup_i A_i$ is an upper bound. Hence, by Zorn's Lemma, Π has a maximal element $(V, \mathfrak{m}_V)$, where $\mathfrak{m}_V \cap R = \mathfrak{m}$.

We show that V is a valuation domain. Let x be an element in K, and suppose $x^{-1} \notin V$. By Lemma 3.4.1, $\mathfrak{m}_V V[x] \neq V[x]$. Take a maximal ideal $\mathfrak{n}$ of $V[x]$, such that $\mathfrak{m}_V V[x] \subseteq \mathfrak{n}$.

Now consider the localization in $\mathfrak{n}$, say S and its maximal ideal $\mathfrak{n}S$. Note that $(S, \mathfrak{n}S) \in \Pi$. Since $V \subseteq S$ and $\mathfrak{m}_V S \subseteq \mathfrak{n}S$, we get that $(V, \mathfrak{m}_V) \leq (S, \mathfrak{n}S)$. As V is a maximal element, $S = V$. Hence $x \in V$. We conclude that V is a valuation domain and $\mathfrak{m}_V \cap R = \mathfrak{m}$. ■

Theorem 3.4.3. *Let R be a Noetherian integral domain and let P be a non-zero prime ideal in R. Then there exists a Noetherian valuation domain V between R and the field of fractions of R such that $\mathfrak{m}_V \cap R = P$.*

Proof. We assume P is the unique maximal ideal of R by localization at P. Take

$$G = gr_P(R) = \bigoplus_{n \geq 0} \left(P^n / P^{n+1}\right) = R/P\left[P/P^2\right].$$

Suppose $P = (f_1, \ldots, f_n)$. Since R/P is a field and P/P^2 is finitely generated, we get that

$$G \cong R/P\left[\bar{f}_1, \ldots, \bar{f}_k\right].$$

In addition, we have the following homomorphism

$$\begin{aligned} \varphi : R/P\left[x_1, \ldots, x_k\right] &\to R/P\left[\bar{f}_1, \ldots, \bar{f}_k\right] \\ x_i &\mapsto \bar{f}_i \end{aligned}$$

Suppose that P/P^2 has only nilpotent elements. Then there exist elements $a_1, \ldots a_k$ such that $\bar{f}_i^{\,a_i} = 0$, $\forall i = 1, \ldots, k$. Thus we have the homomorphism

$$\begin{aligned} \bar{\varphi} : \frac{R/P\left[x_1, \ldots, x_k\right]}{\left(x_1^{a_1}, \ldots, x_k^{a_k}\right)} &\to R/P\left[\bar{f}_1, \ldots, \bar{f}_k\right] \\ \bar{x}_i &\mapsto \bar{f}_i. \end{aligned}$$

Thus by Theorem 2.0.17, $\dim R = 0$. Thus, not every element in P/P^2 is nilpotent, so take $x \in P/P^2$ such that $\bar{x} \in G$ is not nilpotent.

Let $S = R\left[P/x\right] = \left[\frac{f_1}{x}, \ldots, \frac{f_n}{x}\right]$. Note that S is finitely generated as R-

algebra and so, S is Noetherian. Suppose $xS = S$ and write

$$\begin{aligned} 1 =& x\sum_{i=0}^{n} \frac{a_i}{x^i} \text{ for some } a_i \in P,\ i = 1, \ldots, n \\ =& x\left(\sum_{i=0}^{n} \frac{x^{n-i}a_i}{x^n}\right) \text{ with } x^{n-i}a_i \in P^n \\ =& \frac{xa}{x^n} \text{ where } a = \sum_{i=0}^{n} x^{n-i}a_i \\ =& \frac{a}{x^{n-1}} \end{aligned}$$

Then $x^{n-1} = a \in P^n$, which is a contradiction. Hence $xS = PS$ is a proper ideal. By Principal Ideal Theorem, we have that $\dim S_Q = 1$, for every Q prime ideal of S.

Now, if we consider the integral closure T_Q of S_Q, by Lying Over Theorem, there exists a maximal ideal $\mathfrak{n}$ in T containing QT. By Krull-Akizuki Theorem, T_Q is one dimensional, Noetherian and integrally closed. By Theorem 3.3.6, T is a Noetherian valuation domain with maximal ideal $\mathfrak{n}T$. As $Q \subseteq QT \subseteq \mathfrak{n} \subseteq \mathfrak{n}T$, then $Q \subseteq \mathfrak{n}T_{\mathfrak{n}}$. In addition, $xS \subseteq Q$, thus $P \subseteq PS \subseteq \mathfrak{n}T$, and so, $P \subseteq \mathfrak{n}T \cap R$. Finally, take $m \in \mathfrak{n} \cap R$ such that $m \notin P$. Then m is unit in R, and so, m is unit in $T_{\mathfrak{n}}$, which is a contradiction. We conclude that $\mathfrak{n}T_{\mathfrak{n}} \cap R = P$. ■

Chapter 4

Methods in prime characteristic

In this chapter, we study the Frobenius map and how it describes singularities. Specifically, we study F-finiteness, F-splitting, F-regularity, and F-purity. In addition, we give some of their properties, relations among them, and their effects over domains.

Setting 4.0.1. The rings used in this chapter be commutative, with unit and of prime characteristic p.

4.1 Introduction to Frobenius morphism

Definition 4.1.1. *Let R be a ring. The* ***Frobenius morphism*** *is the ring homomorphism*

$$\begin{aligned} F : R &\to R \\ r &\mapsto r^p. \end{aligned}$$

The ***iterated Frobenius*** *is the map $F^e = F \circ \cdots \circ F$ $e > 0$ times. This is,*

$$\begin{aligned} F^e : R &\to R \\ r &\mapsto r^{p^e}. \end{aligned}$$

Proposition 4.1.2. *Let R be a ring. The Frobenius morphism is injective if and only if R is reduced.*

Proof. Let R be reduced. Then F has to be injective, because $x^p = 0$ if and only if $x = 0$.

Now, suppose that the Frobenius map is injective. We proceed by contradiction. Let $x \in R - \{0\}$ be a nilpotent element. Then there exists $\alpha \in \mathbb{N}$ such that $x^\alpha = 0$. In addition, we can find an element $e \in \mathbb{N}$ such that $\alpha < p^e$.

Since F is injective, we get that F^e is also injective. Thus, $F^e(x) = x^{p^e} = 0$, which is a contradiction. ■

Definition 4.1.3. *Let R be a ring and I an ideal. We denote $\mathbf{I^{[p^e]}}$ to the ideal generated by the p^e-powers of all elements of I.*

Setting 4.1.4. Through the rest of this chapter we only consider integral domains.

Proposition 4.1.5. *The Frobenius homomorphism induces the identity map on* $\operatorname{Spec}(R)$*.*

Proof. Consider the following homomorphism

$$\begin{aligned} \phi : \operatorname{Spec}(R) &\to \operatorname{Spec}(R) \\ P &\mapsto F^{-1}(P). \end{aligned}$$

Let $Q \in \operatorname{Spec}(R)$. Since Q is an ideal, we deduce that $F(Q) \subseteq Q$. Hence $Q \subseteq F^{-1}(F(Q)) \subseteq F^{-1}(Q)$.

On the other hand, if $x \in F^{-1}(Q)$, then $x^p \in Q$. Note that $\sqrt{Q} = Q$, and so, $x \in Q$. This is, $\phi(Q) = Q$. ■

Definition 4.1.6. *We define the following algebras.*

- $\mathbf{R^p}$: *the subring of p powers of R. Note that Frobenius factors through the inclusion, this is*

$$R^p \hookrightarrow R.$$

- $\mathbf{F_*R}$: *the ring R using as second operation the restriction of scalars. The elements of this ring are denoted as $\mathbf{F_*r}$. This is equivalent to the algebraic structure given by Frobenius, this is*

$$R \xrightarrow{F} F_*R$$

- $\mathbf{R^{1/p}}$: *the subring of the algebraic closure of* $\operatorname{Frac}(R)$ *whose elements are solutions of the equations $x^p - r = 0$ for each $r \in R$. Note that each* *of these equations have only one solution, because R is of characteristic $p > 0$. The elements of this ring are denoted by* $\mathbf{r^{1/p}}$. *We have the embedding*

$$R \hookrightarrow R^{1/p}.$$

Thus we have the following commutative diagram We have the following commutative diagram of R-algebras

$$\begin{array}{ccc} R^p & \hookrightarrow & R \\ \downarrow & & \downarrow \\ R & \hookrightarrow & F_*R \\ \downarrow & & \downarrow \\ R & \hookrightarrow & R^{1/p}. \end{array}$$

Remark 4.1.7. Note that $F_*R \cong R^{1/p}$, using the R-module homomorphism that sends $F_*r \longmapsto r^{1/p}$. Furthermore, we have that

$$F_*R \cong R^{1/p} \cong R^p \cong R$$

as rings.

In this manuscript we focus on the algebra $F_*^e R$. By Remark 4.1.7, every result about $F_*^e R$ has an equivalent version with the other two algebras.

Definition 4.1.8. *Let R be a ring and M be an R-module. Define* $\mathbf{F_*^e M}$*, for $e > 0$, as the $F_*^e R$-module with operation*

$$\left(F_*^e r\right)\left(F_*^e m\right) = F_*^e\left(rm\right),$$

with $r \in R$, $m \in M$.

Remark 4.1.9. We have that $IF_*^e R = F_*^e I^{[p^e]}$, and $F_*\left(R/m^{[p]}\right) \cong F_*R/F_*m^{[p]}$.

Example 4.1.10. Let $R = \mathbb{F}_p\left[x_1, ..., x_n\right]$. Note that

$$F_*R = \mathbb{F}_p\left[F_*x_1^{\beta_1}, \ldots, F_*x_n^{\beta_n}\right],$$

because $a = a^p$ $\forall a \in \mathbb{F}_p$. Now, using the division algorithm, $\beta_i = q_i p + \alpha_i$ with $0 \leq \alpha_i \leq p - 1$. Thus, for every monomial in F_*R, we have that

$$\begin{aligned} a_{(\beta_1,\ldots,\beta_n)} F_*x_1^{\beta_1} \cdots F_*x_n^{\beta_n} =& a_{(\beta_1,\ldots,\beta_n)} F_*x_1^{q_1 p+\alpha_1} \cdots F_*x_n^{q_n p+\alpha_n} \\ =& a_{(\beta_1,\ldots,\beta_n)} x_1^{q_i} \cdots x_n^{q_n} F_*x_1^{\alpha_1} \cdots F_*x_n^{\alpha_n}, \end{aligned}$$

which belongs to $R\left[F_*x_1^{\alpha_1}\cdots F_*x_n^{\alpha_n}\right]$. Note that this monomial belongs uniquely to $R\left[F_*x_1^{\alpha_1}\cdots F_*x_n^{\alpha_n}\right]$ because of the choose of α_i. We conclude F_*R is a free R-mod with basis

$$\left\{F_*x_1^{\alpha_1},\ldots,F_*x_n^{\alpha_n}|0\leq\alpha_i\leq p-1\right\}.$$

The following theorem shows that F_*R commutes with localization.

Proposition 4.1.11. *Let R be a ring and $W\subseteq R$ be any multiplicative system. Then $W^{-1}F_*R\cong F_*\left(W^{-1}R\right)$.*

Proof. Consider the map

$$\begin{aligned}\varphi: W^{-1}F_*R &\to F_*\left(W^{-1}R\right)\\ \frac{1}{g}F_*r &\mapsto F_*\left(\frac{r}{g^p}\right).\end{aligned}$$

First we show that φ is well-defined. Take two related elements, say $\frac{1}{g}F_*r\sim\frac{1}{h}F_*s$. Note that $\frac{1}{g}$ and $\frac{1}{h}$ may be thought as scalars for F_*R. Then we have that $F_*\frac{r}{g^p}\sim F_*\frac{s}{h^p}$. Note that φ is a homomorphism.Indeed

$$\frac{1}{g}F_*r+\frac{1}{h}F_*s=\frac{1}{hg}F_*\left(h^pr+g^ps\right).$$

Thus,

$$F_*\left(h^pr+g^ps\right)=F_*\left(\frac{r}{g^p}+\frac{s}{h^p}\right)=F_*\frac{r}{g^p}+F_*\frac{s}{h^p}.$$

In addition, if $\alpha\in R$, then

$$\alpha\left(\frac{1}{g}F_*r\right)=\frac{1}{g}F_*\alpha^pr.$$

Therefore

$$F_*\frac{\alpha^pr}{g^p}=\alpha F_*\frac{r}{g^p}.$$

Now, let $\frac{1}{g}F_*r\in\ker\varphi$. Then, there exists $s\in W$ such that $sr=0$. Since we are in a domain, $r=0$, and so, $\frac{1}{g}F_*r=0$. Furthermore, for any $F_*\frac{r}{g}$, we have the element $\dfrac{1}{g}F_*rg^{p-1}$, such that

$$\varphi\left(\frac{1}{g}F_*rg^{p-1}\right) = F_*\frac{r}{g}.$$

Hence φ is an isomorphism.

■

Theorem 4.1.12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let $\widehat{R}$ be the completion at $\mathfrak{m}$. Then there is a identification of the maps $\widehat{R} \to \widehat{F_*R}$ and $\widehat{R} \to F_*\widehat{R}$.*

Proof. We have that $\widehat{F_*R} = \lim_{\leftarrow} F_*R/\mathfrak{m}^n\,(F_*R)$. Thus, the morphism

$$R/\mathfrak{m}^n \to F_*R/\mathfrak{m}^n\,(F_*R)$$

induces the morphism

$$\widehat{R} \to \widehat{F_*R}.$$

On the other hand, we know that $\forall n \in \mathbb{N}$

$$F_*R/\mathfrak{m}^n F_*R \cong F_*R/F_*\left((\mathfrak{m}^n)^{[p]}\right) \cong F_*\left(R/\,(\mathfrak{m}^n)^{[p]}\right).$$

In addition, $\left\{(\mathfrak{m}^n)^{[p]}\right\}_n$ and $\{\mathfrak{m}^n\}_n$ are confinal. Indeed, $(\mathfrak{m}^n)^{[p]} \subseteq \mathfrak{m}^n$ for every $n \in \mathbb{N}$. Now, let $x_1, \ldots, x_d$ be the generators of $\mathfrak{m}$. Note that $\mathfrak{m}^{pd} = \langle x_1^{\alpha_1} \cdots x_d^{\alpha_d} \mid \alpha_1 + \cdots + \alpha_d = pd\rangle$, so there exists i such that $\alpha_i > p$ in each generator of $\mathfrak{m}^{pd}$. This is, $\mathfrak{m}^{pd} \subseteq \mathfrak{m}^{[p]}$, and so, $\mathfrak{m}^{pdn} \subseteq \left(\mathfrak{m}^{[p]}\right)^n$ for every $n \in \mathbb{N}$.

Hence,

$$\lim_{\leftarrow} F_*R/\mathfrak{m}^n F_*R = \lim_{\leftarrow} F_*R/F_*\left((\mathfrak{m}^n)^{[p]}\right) = \lim_{\leftarrow} F_*\left(R/\,(\mathfrak{m}^n)^{[p]}\right)$$

Thus we have the map

$$\widehat{R} \to F_*\widehat{R}.$$

■

The module F_*^eR gives us information about both the domain and its residue field. The following theorems are examples of this fact.

Proposition 4.1.13. *Let $(R, \mathfrak{m}, K)$ be an local domain. If F_*R is a finitely generated module, then it is minimally generated by*

$$[K : K^p] \dim_K \left(R/\mathfrak{m}^{[p]}\right)$$

elements.

Proof. By Theorem 2.0.20, the minimal number of generators is the dimension of $F_*R/\mathfrak{m}F_*R$. Since $\mathfrak{m}F_*R \cong F_*\mathfrak{m}^{[p]}$, we have that $F_*R/\mathfrak{m}F_*R \cong F_*R/F_*\mathfrak{m}^{[p]} \cong F_*\left(R/\mathfrak{m}^{[p]}\right)$.

Hence $\dim_{F_*K} F_*\left(R/\mathfrak{m}^{[p]}\right) = \dim_K \left(R/\mathfrak{m}^{[p]}\right)$. Since we have the field extension $K \subseteq F_*K$, we conclude that

$$\dim_K (F_*R) = [K : K^p] \dim_K \left(R/\mathfrak{m}^{[p]}\right).$$

■

Theorem 4.1.14 (Kunz's Theorem)**.** *If R is a Noetherian domain, then R is regular if and only if F_*R is a flat R-module.*

Proof. In this work we only show that if R is regular then F_*R is flat as R-module. For the complete proof, we refer to the paper "Characterizations of regular local rings of characteristic p" by Ernst Kunz[Kun69]. Note that R is regular if and only if $R_\mathfrak{m}$ $\forall \mathfrak{m} \in \text{Max}(R)$ and F_*R is flat over R if and only if $(F_*R)_\mathfrak{m} \cong F_*R_\mathfrak{m}$ is flat over $R_\mathfrak{m}$. Hence, we focus on the local case.

First we prove that R regular implies that F_*R is flat. Let $(R, \mathfrak{m}, K)$ be a regular local domain. By the Cohen Structure Theorem, we have

$$R \cong K[\![x_1, ..., x_n]\!].$$

By Remark 4.1.7, $K[\![F_*x_1, ..., F_*x_n]\!]$ is a free $K[\![x_1, ..., x_n]\!]$-module, thus

$$K[\![x_1, ..., x_n]\!] \subseteq K[\![F_*x_1, ..., F_*x_n]\!]$$

is a flat extension. Since F_*K is a flat K-module, $F_*K \otimes_K K[\![F_*x_1, ..., F_*x_n]\!]$ is also flat. By properties of extension of scalars, we have that $F_*K \otimes_K K\left[F_*x_1, ..., F_*x_n\right] \cong F_*K\left[F_*x_1, ..., F_*x_n\right]$, Thus we have that

$$\begin{aligned} \frac{F_*K \otimes_K K\left[F_*x_1, ..., F_*x_n\right]}{\left(F_*x_1, ..., F_*x_n\right)^j} &\cong \frac{F_*K\left[F_*x_1, ..., F_*x_n\right]}{\left(F_*x_1, ..., F_*x_n\right)^j} \\ \Rightarrow \lim_{\leftarrow} \frac{F_*K \otimes_K K\left[F_*x_1, \ldots, F_*x_n\right]}{\left(F_*x_1, \ldots, F_*x_n\right)^j} &\cong \lim_{\leftarrow} \frac{F_*K\left[F_*x_1, \ldots, F_*x_n\right]}{\left(F_*x_1, \ldots, F_*x_n\right)^j} \\ \Rightarrow F_*K \widehat{\otimes}_K K[\![F_*x_1, \ldots, F_*x_n]\!] &\cong F_*K[\![F_*x_1, \ldots, F_*x_n]\!]. \end{aligned}$$

We conclude that F_*R is flat for R, because a composition of flat maps is flat.

■

Now that we know some properties from the Frobenius homomorphism, we are ready to introduce the first F-singularity.

Definition 4.1.15. *We say that a domain R is F**-finite** if F_*R is a finitely generated R-module.*

Remark 4.1.16. Quotients and localizations of a F-finite ring, are also F-finite. In addition, any finitely generated algebra over a F-finite ring is F-finite.

4.2 F-finiteness in Excellence Rings

In this section we prove an equivalence from the main result.

Definition 4.2.1. *Let R be a ring.*

- *We say that R is a **Grothendieck ring** or a **G-ring** if it is Noetherian and for every $P \in \operatorname{Spec}(R)$,*

$$R_P \to \widehat{R_P}$$

  is regular.

- *We say that R is a **J-2 ring** if for every finitely generated R-algebra S, the singular points of $\operatorname{Spec}(S)$ form a closed subset.*

- *We say that R is **universally catenary** if every finitely generated R-algebra are catenary rings.*

- *We say that R is **excellent** if it is a universally catenary, J-2, G-ring.*

Example 4.2.2. Some examples of excellent rings are

- fields,
- complete Noetherian rings
- $\mathbb{C}[\![x_1, \ldots, x_n]\!]$, $\mathbb{R}[\![x_1, \ldots, x_n]\!]$

Remark 4.2.3. If R is an excellent ring and W any multiplicative set, then R_W is also excellent. In addition, fnitely generated algebras are also excellent, for instance the coordinate ring of a variety over $\mathbb{C}$ and $\mathbb{R}$.

In order to prove the equivalence, we mention the following lemma.

Lemma 4.2.4. *Let A be an excellent domain. The integral closure of A in any finite extension of its fraction field is finite as a A-module.*

Theorem 4.2.5. *Let R be a Noetherian domain. Then R is F-finite if and only if it is excellent and its fraction field is F-finite.*

Proof. First, suppose R is F-finite and $K = \mathrm{Frac}(R)$. Then R is a finitely generated R^p-module. Note that

$$\begin{aligned} R \otimes_{R^p} K^p \cong & R \otimes_{R^p} \left(R_{(0)}\right)^p \\ \cong & R \otimes_{R^p} R^p_{(0)^p} \\ \cong & R_{(0)^p} \\ \cong & R_{(0)} \\ = & K. \end{aligned}$$

As K^p is a R^p-module, it is finitely generated. Thus, $R \otimes_{R^p} K^p$ is finitely generated and R^p-module. Note R^p is a K^p-module, and so, $K = R \otimes_{R^p} K^p$ is a finitely generated K^p-module. We recall that F-finite Noetherian rings are excellent [Kun76, Theorem 2.5] .

Now, as $R^p \cong R$ as rings, then R^p is also an excellent ring. On the other hand, K is a finitely generated K^p-module, by Lemma 4.2.4, the integral closure of R^p in K is a finitely generated R^p-module. Let $r \in R$. Then we have the polynomial $f(x) = x - r$ in $K[x]$, such that $f(r) = 0$. Hence, R is contained in the integral closure of R^p. Since R^p is Noetherian, we conclude that R is a finitely generated R^p-module. ■

4.3 F-split domains

The usual definition of "a splitting map" can be specialized to the Frobenius map. This brings the following definition.

Definition 4.3.1. *Let R be a domain. We say that R is **Frobenius split**, or F**-split** if there is a map*

$$\varphi : F_* R \to R$$

such that $\varphi \circ F = Id_R$.

Remark 4.3.2. Saying that R is F-split is equivalent to the following:

- there exists $\pi \in \operatorname{Hom}(F_* R, R)$ such that $\pi(F_* 1) = 1$, and
- $F_* R \cong R \oplus M$, with M an R-module.

The following definitions are examples of F-split rings.

Definition 4.3.3. *Let K be a field, $S = [x_1, \ldots, x_n]$ be a polinomial ring and $I \cap_{i=1}^{n} P_i$ be an square free monomial ideal, where each $P_i = (\{x_j \mid j \in S_i\})$ is a monomial prime ideal. Then the ring R/I is called a **Stanley-Reisner ring**.*

Definition 4.3.4.

1. **(Generic)** *If $X = (x_{i,j})$ is an $m \times r$ matrix of variables, then $I_t(X)$ is the ideal of $R = k[X]$ generated by the t-minors of X.*

2. **(Symmetric)** *If $Y = (y_{i,j})$ is an $m \times m$ generic symmetric matrix, i.e., $y_{i,j} = y_{j,i}$ for every $1 \leq i, j \leq m$, then $I_t(Y)$ is the ideal of $R = k[Y]$ generated by the t-minors of Y. The minors $[i_1, \ldots, i_t | j_1, \ldots, j_t]$ such that $i_s \leq j_s$ for every $1 \leq s \leq t$ are called doset minors and they generate $I_t(Y)$.*

3. **(Skew-symmetric)** *Let $Z = (z_{i,j})$ be an $m \times m$ generic skew symmetric matrix, i.e., $z_{i,j} = -z_{j,i}$ for every $1 \leq i < j \leq m$, and $z_{i,i} = 0$ for every $1 \leq i \leq m$. The minors of the form $[i_1, \ldots, i_{2t}] := [i_1, \ldots, i_{2t} | i_1, \ldots, i_{2t}]$ are squares of certain polynomials of $R = k[Z]$. These polynomials are called the Pfaffians of Z. The ideal $P_{2t}(Z)$ is the one generated by the $2t$-Pfaffians of Z.*

Remark 4.3.5. Let $c \in R$ be a nonzero element with $e \in \mathbb{N}$ and $\pi \in \operatorname{Hom}(F_*^e R, R)$ such that $\pi(F_*^e c) = 1$. Then take $\theta = \pi \circ \times F_*^e c \circ F^{e-1}$, where

$\times F^e_* c$ is the map defined by $r \mapsto r F^e_* c$. We have that

$$\begin{aligned}\theta(F_* 1) &= \left(\pi \circ \times F^e_* c \circ F^{e-1}\right)(F_* 1) \\ &= (\pi \circ \times F^e_* c)(F^e_* 1) \\ &= \pi(F^e_* c) \\ &= 1\end{aligned}$$

This implies that R is F-split.

Proposition 4.3.6. *If $(R, \mathfrak{m})$ is a F-finite regular local domain, then R is F-split.*

Proof. Note that, by Theorem 4.1.14, $F_* R$ is flat and finitely generated, and so, it is also free. Thus, any minimal set of generators is a free basis for $F_* R$. By Theorem 2.0.20, we find one by choosing a basis for $F_* R / \mathfrak{m} F_* R = F_* \left(R/\mathfrak{m}^{[p]}\right)$. We take the element $F_* 1$ as part of a basis for $F_* R$, and consider the projection

$$\begin{aligned}\pi : (F_* 1) R \oplus (F_* b_1) R \cdots \oplus (F_* b_j) R &\to R \\ (F_* 1) r_0 \oplus (F_* b_1) r_1 \cdots \oplus (F_* b_j) r_j &\mapsto r_0.\end{aligned}$$

Finally, we have that

$$\begin{aligned}(\pi \circ F)(1) &= \pi(1^p) \\ &= \pi(F_* 1) \\ &= 1.\end{aligned}$$

We conclude that R is F-split. ■

Proposition 4.3.7. *A domain R is F-split if and only if the module $R/\operatorname{Im}\psi$ is zero, where ψ is defined as*

$$\begin{aligned}\psi : \operatorname{Hom}(F_* R, R) &\to R \\ \phi &\mapsto \phi(F_* 1).\end{aligned}$$

Proof. We first assume that R is F-split. Then there exists $\pi \in \operatorname{Hom}(F_* R, R)$ such that $\pi \circ F = Id$. By Theorem 2.0.25, ψ is surjective. Hence $R/\operatorname{Im}\psi$ is zero.

Conversely, if $R/\operatorname{Im}\psi$ is zero, then $\operatorname{Im}\psi = R$. Appliying the Theorem 2.0.25, we get that the Frobenius map splits.

■

Corollary 4.3.8. *A domain R is F-split if and only if $R_{\mathfrak{m}}$ is also F-split $\forall \mathfrak{m} \in \operatorname{Max}(R)$.*

Proof. This follows from Proposition 4.3.7 and Theorem 2.0.23. ■

4.4 F-regular domains

We introduce another class of rings closely related to the F-splitting.

Definition 4.4.1. *Let R be a F-finite domain. We say that R is F**-regular** if for every $c \neq 0$ there exists $e \in \mathbb{N}$ such that the R-module map*

$$\begin{aligned} R &\to F_*^e R \\ 1 &\mapsto F_*^e c. \end{aligned}$$

splits as a map of R-modules. This is, there exist $e \in \mathbb{N}$ and $\pi \in \operatorname{Hom}(F_*^e R, R)$ such that $\pi(F_*^e c) = 1$.

This class of rings was introduced by Hochster and Huneke [HH89a] with the name "strongly F-regularity" along with other related notions. We recall that Datta and Smith [DS16] called it "F-split regularity". Throughout this work we simple called it F-regularity.

Theorem 4.4.2 ([AL03])**.** *Let R be a F-finite domain. Then R is F-regular if and only if*

$$\lim_{e\to\infty} \frac{\operatorname{free.rank} F_*^e R}{\operatorname{rank} F_*^e R} > 0.$$

The limit of the Theorem 4.4.2 is called ***F*-signature**. It first appeared implicity in the work of of Smith and Van den Bergh [SVdB97]. Later Huneke and Leuschke [HL02] coined the term if the limitexist. The convergence of F-signature was proven by Tucker [Tuc12].

Now we give some examples of F-regular rings.

Example 4.4.3.

- Determinantal varieties,
- Cluster-algebras [BMRS15]
- Invariant rings [HH89b]

- Toric rings [HH89b]

Remark 4.4.4. Note that if we consider $c = 0$ in the Definition 4.4.1, then the map

$$\begin{aligned} \varphi : R &\to F^e_* R \\ 1 &\mapsto F^e_* c, \end{aligned}$$

splits. Hence, there exists $\pi : F^e_* R \to R$ such that $\pi\left(F^e_* c\right) = 1$. Thus,

$$1 = \pi\left(F^e_* 0\right) = \pi\left(r F^e_* 0\right) = \left(\pi \circ \varphi\right)(r) = r,$$

$\forall r \in R$. In particular, take $r = 0$. Then $1 = 0$, this is $R = \{0\}$.

Remark 4.4.5. Let R be a F-regular Noetherian domain and $c = 1$. There exist $e \in \mathbb{N}$ and $\pi \in \operatorname{Hom}\left(F^e_* R, R\right)$ such that $\pi\left(F^e_* 1\right) = 1$. Note that if $e = 1$, we are done. Now, consider the homomorphism $\theta = \pi \circ F^{e-1}$. Then we have

$$\begin{aligned} \theta\left(F_* 1\right) &= \left(\pi \circ F^{e-1}_*\right)\left(F_* 1\right) \\ &= \pi\left(F^e_* 1\right) \\ &= 1 \end{aligned}$$

Therefore R is F-split.

Theorem 4.4.6. *If $(R, \mathfrak{m})$ is a Noetherian local F-finite regular domain, then R is F-regular.*

Proof. Take $c \neq 0$ in R. There exists $e \in \mathbb{N}$ such that $c \notin \mathfrak{m}^{[p^e]}$, by Theorem 2.0.15. Thus $F^e_* c \notin F^e_* \mathfrak{m}^{[p^e]}$. By Theorem 2.0.20, $F^e_* c$ is part of a generating set for $F^e_* R$ as R-module. By Theorem 4.1.14, $F^e_* R$ is a free R-module. Let $\{F^e_* c, b_1, \ldots b_j\}$ be a basis. Consider the projection

$$\begin{aligned} \pi : \left(F_* c\right) R \oplus \left(F_* b_1\right) R \cdots \oplus \left(F_* b_j\right) R &\to R \\ \left(F_* c\right) r_0 \oplus \left(F_* b_1\right) r_1 \cdots \oplus \left(F_* b_j\right) r_j &\mapsto r_0. \end{aligned}$$

Note that $\pi\left(F^e_* c\right) = 1$. Thus R is F-regular.

■

Proposition 4.4.7. *Let $i : S \hookrightarrow R$ be an inclusion of Noetherian domains that splits as S-modules. If R is F-regular, then S is also F-regular.*

Proof. We have a homomorphism Φ such that $\Phi \circ i = Id_S$. In addition, since R is regular, $\forall c \in R - \{0\}$, there exists $e \in \mathbb{N}$ and $\pi_c \in \operatorname{Hom}(F_*^e R, R)$ such that $\pi_c (F_*^e c) = 1$. Take $d \in S - \{0\}$ and $c = i(d)$. Consider the map $\theta_d = \Phi \circ \pi_c \circ F_*^e i$. We have that

$$\begin{aligned}
\theta_d (F_*^e d) &= (\Phi \circ \pi_c \circ F_*^e i)(F_*^e d) \\
&= (\Phi \circ \pi_c \circ)(F_*^e c) \\
&= \Phi (1_R) \\
&= 1_S.
\end{aligned}$$

We conclude that S is F-regular. ∎

Hence, we note the strong relation between this two notions of F-splitness and F-regularity. This yields to the following definition.

Definition 4.4.8. *Let R be an F-finite domain, and c be a non-zero element. We say that R is **eventually F-split along** c if there exists $e \in \mathbb{N}$ such that*

$$\begin{aligned}
R &\to F_*^e R \\
1 &\mapsto F_*^e c
\end{aligned}$$

splits.

Note that, R is F-regular if and only if it is F-split along every non-zero element.

Remark 4.4.9. Let R be eventually F-split along some c, and $d \in R$ such that $c = dh$ for some h. Thus there exist $e \in \mathbb{N}$ and $\pi \in \operatorname{Hom}(F_*^e R, R)$ such that $\pi (F_*^e c) = 1$.

Consider the following map

$$\begin{aligned}
\theta : F_*^e R &\to R \\
F_*^e r &\mapsto \pi (F_*^e r \cdot F_*^e h).
\end{aligned}$$

Since $\theta (F_*^e d) = 1$, we have that R is eventually F-split along d. Since R is commutative, R is also F-split along h.

Theorem 4.4.10. *Let R be an F-finite Noetherian domain.*

a) If R is F-regular, then so is R_W for any multiplicative system W.

b) If $R_{\mathfrak{m}}$ is F-regular for every $\mathfrak{m} \in \operatorname{Max}(R)$, then R is F-regular.

Proof.

a) Let $\frac{c}{w} \in R - \{0\}$. We show that R_W is eventually F-split along $\frac{c}{w}$. It suffices to prove it for $\frac{c}{1}$, as $\frac{c}{w} = \frac{c}{1} \times \frac{1}{w}$.

Note that c is a non-zero element, thus there exist $e \in \mathbb{N}$ and $\pi \in \operatorname{Hom}(F_*^e R, R)$ such that $\pi(F_*^e c) = 1$. Consider the homomorphism

$$\begin{aligned}\pi_W : F_*^e R_W \to & R_W \\ \frac{F_*^e r}{1} \mapsto & \frac{\pi(r)}{1}.\end{aligned}$$

We have that $\pi_W\left(\frac{F_*^e c}{1}\right) = \frac{\pi(F_*^e c)}{1} = \frac{1}{1}$. Thus, R_W is F- split along $\frac{c}{1}$. We conclude that it is F-regular.

b) Take $c \neq 0$. By Theorem 2.0.25, for each maximal ideal $\mathfrak{m}$ the map

$$\begin{aligned}\psi_{e_{\mathfrak{m}}} : \operatorname{Hom}(F_*^{e_{\mathfrak{m}}} R, R)_{\mathfrak{m}} \to & R_{\mathfrak{m}} \\ \phi \mapsto & \frac{\phi(F_*^{e_{\mathfrak{m}}} c)}{1}.\end{aligned}$$

is surjective for any $e_{\mathfrak{m}} \gg 0$. Fix $\mathfrak{m} \in \operatorname{Max}(R)$. We have that, $(R/\operatorname{Im}\psi_{e_{\mathfrak{m}}})_{\mathfrak{m}} \cong R_{\mathfrak{m}}/(\operatorname{Im}\psi_{e_{\mathfrak{m}}})_{\mathfrak{m}} = 0$.

Let $A_{e_{\mathfrak{m}}} = R/\operatorname{Im}\psi_{e_{\mathfrak{m}}}$ and $P \in \operatorname{Ass}(A_{e_{\mathfrak{m}}})$. Then we have that

$$\begin{aligned}R/P \hookrightarrow A_{e_{\mathfrak{m}}} \Rightarrow & (R/P)_{\mathfrak{m}} \hookrightarrow (A_{e_{\mathfrak{m}}})_{\mathfrak{m}} = 0 \\ \Rightarrow & (R/P)_{\mathfrak{m}} = 0 \\ \Rightarrow & P \nsubseteq \mathfrak{m}.\end{aligned}$$

Hence, $P \cap R\ \mathfrak{m} \neq \emptyset$, $\forall P \in \operatorname{Ass}(A_{e_{\mathfrak{m}}})$. Consider an element in this intersection, f_P. Since $A_{e_{\mathfrak{m}}}$ is a finitely generated R-module, it has finitely many associated primes, so we take the element

$$f_{\mathfrak{m}, e_{\mathfrak{m}}} := \prod_{P \in \operatorname{Ass}(A_{e_{\mathfrak{m}}})} f_P$$

which is not in $\mathfrak{m}$. Moreover, $\text{Ass}\left(A_{e_{\mathfrak{m}}}\right)_f = 0$, because for every $P \in \text{Ass}\left(A_{e_{\mathfrak{m}}}\right)$ we have that $P \cap \left\{f^n_{\mathfrak{m},e_{\mathfrak{m}}} | n \in \mathbb{N}\right\} = \emptyset$. Hence $\left(A_{e_{\mathfrak{m}}}\right)_{f_{\mathfrak{m},e_{\mathfrak{m}}}} = 0$. Note that we can get elements $e_{\mathfrak{m}}$ and $f_{\mathfrak{m},e_{\mathfrak{m}}}$ for each maximal ideal $\mathfrak{m}$. Thus, we consider

$$U_{f_{\mathfrak{m}},e_{\mathfrak{m}}} = \{P \in \text{Spec}\,(R)\,|\,f_{\mathfrak{m},e_{\mathfrak{m}}} \notin P\}$$

which are basic open sets in the Zariski topology, and the open set

$$\cup_{\mathfrak{m}\in \text{Max}(R)} U_{f_{\mathfrak{m}}}.$$

Since $\text{Spec}\,(R)$ is quasi-compact and this union is a cover for it, there exists a finite subcover

$$U_{f_{\mathfrak{m}_1}}, \ldots, U_{f_{\mathfrak{m}_j}}.$$

On the other hand, $\forall \mathfrak{m} \in \text{Max}\,(R)$, we have an element $e_{\mathfrak{m}} \gg 0$ such that the map $\psi_{e_{\mathfrak{m}}} : \text{Hom}\left(F^{e_{\mathfrak{m}}}_* R, R\right)_{\mathfrak{m}} \to R_{\mathfrak{m}}$ is surjective.

Consider

$$\hat{e} = \max\left\{e_{\mathfrak{m}_1}, \ldots e_{\mathfrak{m}_j}\right\}.$$

Hence, $\forall \mathfrak{m} \in \text{Max}\,(R)$ the map $\psi_{\hat{e}} : \text{Hom}\left(F^{\hat{e}}_* R, R\right)_{\mathfrak{m}} \to R_{\mathfrak{m}}$ is surjective. By the Theorem 2.0.24, the map

$$\begin{aligned} \psi : \text{Hom}\left(F^{\hat{e}}_* R, R\right) &\to R \\ \phi &\mapsto \phi\left(F^{\hat{e}}_* c\right), \end{aligned}$$

is surjective. Thus, we obtain a map

$$\begin{aligned} \phi : F^{\hat{e}}_* R &\to R \\ F^{\hat{e}}_* c &\mapsto 1. \end{aligned}$$

On the other hand, since $R_{\mathfrak{m}}$ is F-regular, each $R_{\mathfrak{m}}$ is F-split, by Remark 4.4.5. By Corollary 4.3.8, R is F-split. Therefore, we can repeatedly compose ϕ with a Frobienius splitting in order to get a splitling for a larger $e \in \mathbb{N}$. We conclude R is F-regular.

■

Furthermore, under certain conditions, F-regularity is preserved by localization at an element.

Theorem 4.4.11. *Let R be an F-finite Noetherian domain. Suppose that $d \in R$ is such that R_d is F-regular. If there exist $e \in \mathbb{N}$ and an R-module map*

$$\begin{aligned} \pi : F_*^e R &\to R \\ F_*^e d &\mapsto 1, \end{aligned}$$

then R is F-regular.

Proof. Note that R is F-split along d because there exists π which is the splitting of the morphism

$$\begin{aligned} \varphi : R &\to F_*^e R \\ 1 &\mapsto F_*^e d. \end{aligned}$$

By Remark 4.3.5, R is F-split.

Take $c \neq 0$ and the homomorphism

$$\begin{aligned} \psi_f : \operatorname{Hom}\left(F_*^f R, R\right) &\to R \\ \phi &\mapsto \phi\left(F_*^f c\right) \end{aligned}$$

for some $f \gg 0$. Since R_d is F-regular, we have that $(\psi_f)_d$ is surjective. Hence we have that $\psi_f \otimes R_d$ is surjective. Therefore, there exists and element such that

$$\begin{aligned} &(\psi_f \otimes R_d)\left(\sum_{i=1}^{n}\left(\kappa_i \otimes \frac{r_i}{d^{m_i}}\right)\right) = 1 \text{ with } r_i \in R \text{ and } k_i \in F_*^f R \\ \Rightarrow &\sum_{i=1}^{n}\left(\psi_f(\kappa_i) \otimes \frac{r_i}{d^{m_i}}\right) = 1 \\ \Rightarrow &\sum_{i=1}^{n}\left(\frac{r_i}{d^{m_i}}\psi_f(\kappa_i)\right) = 1 \\ \Rightarrow &\frac{\sum_{i=1}^{n}\left(r_i d^{m-m_i}\psi_f(\kappa_i)\right)}{d^m} = 1 \text{ where } m = \max\{m_i \mid i = 1, \ldots, n\} \\ \Rightarrow &\sum_{i=1}^{n}\left(r_i d^{m-m_i}\psi_f(\kappa_i)\right) = d^m \\ \Rightarrow &\psi_f\left(\sum_{i=1}^{n} r_i d^{m-m_i}\kappa_i\right) = d^m. \end{aligned}$$

This is, $d^m \in \operatorname{Im} \psi_f$, for some m. Thus, there exists $\phi \in \operatorname{Hom}\left(F_*^f R, R\right)$ such that $\phi\left(F_*^f c\right) = d^m$. Without loss of generality we assume that $m = p^t$ with $t \in \mathbb{Z}$.

Let $\theta : F_*^t R \to R$ be a F-splitting. Note that $\pi \circ F_*^e \theta \circ F_*^{t+e} \phi$ gives a splitting for

$$R \to F_*^{e+t+f} R$$
$$1 \mapsto F_*^{e+t+f} c.$$

Thus R is F-split along c. Since c was an arbitrary element, we conclude that R is F- regular.

■

Proposition 4.4.12. *Let $(R, \mathfrak{m})$ be a F-finite Noetherian domain. Then R is F-regular if and only if the completion $\widehat{R}$ at $\mathfrak{m}$ is F-regular.*

Proof. Note that if we take $c \neq 0$, then the image of c in $\widehat{R}$ is not zero. Hence, for every $e \in \mathbb{N}$ we have a map

$$\psi : \operatorname{Hom}\left(F_*^e R, R\right) \to R$$
$$\phi \mapsto \phi\left(F_*^e c\right)$$

which is surjective if and only if the map $\psi \otimes \widehat{R}$ remains surjective, by faithfulness of $\widehat{R}$.

Suppose that $\widehat{R}$ is F-regular. Then ψ is surjective for a large e. Thus R is F-regular.

Now, suppose R is F-regular. We now show that there exists $c \in R$ such that $\widehat{R_c}$ is regular. Since R is a F-finite local domain, F_*R is torsion free. Then $F_*R \otimes K \cong K^\alpha$ for some $\alpha \in \mathbb{N}$. Let $\{v_1, \ldots, v_t\}$. We have te maps

$$F_*R \hookrightarrow F_*R \otimes K \longleftrightarrow K^\alpha$$

$$v_i \hookrightarrow v_i \otimes 1 \longleftrightarrow \frac{a_{1,i}}{b_{1,i}} e_1 + \cdots + \frac{a_{\alpha,i}}{b_{\alpha,i}} e_\alpha$$

This is, $F_*R \xrightarrow{\varphi} K^\alpha$ is an inclusion. Let $b = \prod_{s=1,i=1}^{\alpha,t} b_{s,i}$. Note that $b\varphi : F_*R \to R^\alpha$ is also an inclusion, and so is $b\varphi_b : (F_*R)_b \to R_b^\alpha$. Thus, there exists b such that $(F_*R)_b$ is a free module. We have that $\widehat{(F_*R)_b} \cong \widehat{R} \otimes (F_*R)_b$ is a free R-module. Hence $\left(\widehat{(F_*R)_b}\right)_P$ is free, $\forall P \in \operatorname{Spec}(R)$. We conclude that $\widehat{(F_*R)_b}$ is regular, so it is F-regular.

■

Remark 4.4.13. Let R be a Noetherian domain whose regular locus is open. Note that the regular locus is not empty. Indeed, consider the prime ideal 0. Then R_0 is a zero-dimensional local domain, thus it is a field and so, it is regular.

The regular locus is open, therefore it is the complement of closed set

$$V(I) = \{Q \in \operatorname{Spec}(R) \,|\, I \subseteq Q\},$$

for some ideal I. Note that I has height at least one. Now, since $I \nsubseteq P$, for any minimal prime P, by Prime Avoidence Theorem, we choose $c \in I$ such that R_c is regular.

Proposition 4.4.14. *If R is a Noetherian domain whose regular locus is open, then there is an element $c \neq 0$ such that R_c is regular.*

Proof. First, note that $R_{(0)}$ is a field, and so it is regular. This is, the regular locus is nor empty. Thus, there exists and ideal I, such that the regular locus of R is $V(I)^C$. Let $c \in I \setminus \{0\}$. We have that

$$V(\{c\}) \subseteq V(I).$$

Therefore $\operatorname{Spec}(R_c) \subseteq V(I)^C$. ■

Theorem 4.4.15. *Let $(R, \mathfrak{m})$ be a F-finite Noetherian domain. The locus points $P \in \operatorname{Spec}(R)$ where R_P is F-regular is open.*

Proof. Since R is F-finite, the regular locus is open. By Remark 4.4.13, there exists $d \neq 0$ such that R_d is regular, thus F-regular. Note that for every $g \in R$ we have that $(R_d)_g$ is F-regular, because it is the localization of a regular ring. Hence $(R_g)_d$ is also F-regular.

Consider the map

$$\begin{aligned} \psi : \operatorname{Hom}(F_*^e R, R) &\to R \\ \phi &\mapsto \phi(F_*^e d). \end{aligned}$$

Let $\mathfrak{m}$ be in the F-regular locus of R. Then $R_\mathfrak{m}$ is F-regular and the morphism

$$\begin{aligned} \psi_\mathfrak{m} : (\operatorname{Hom}(F_*^e R, R))_\mathfrak{m} &\to R_\mathfrak{m} \\ \phi_\mathfrak{m} &\mapsto \phi(F_*^e d)_\mathfrak{m} \end{aligned}$$

is surjective for some $e \gg 0$. Thus $\mathfrak{m} \notin \operatorname{Supp}(R/\operatorname{Im}\psi)$, and $\operatorname{Im}\psi \not\subseteq \mathfrak{m}$. Take $g \in \operatorname{Im}\psi \setminus \mathfrak{m}$. Then there exists $\phi \in \operatorname{Hom}(F_*^e R, R)$ such that $\phi(F_*^e d) = g$. Hence we have

$$\begin{aligned} \phi_g : F_*^e R_g &\to R_g \\ (F_*^e d)_g &\mapsto 1. \end{aligned}$$

By Theorem 4.4.11, R_g is F-regular.

Note that $P_g \in \operatorname{Spec}(R_g)$. Then g is not in the contraction of P_g in R. Consider

$$U = \left\{ P \in \operatorname{Spec}(R) \,|\, P = P_g^c \text{ for some } P_g \in \operatorname{Spec}(R_g) \right\}.$$

Since U is the complement of a closed set in R, it is open and $\mathfrak{m} \in U$. Now, we show that R_P is F-regular $\forall P \in U$.

Since $(R_g)_P \cong R_P$ and R_g is F-regular, by Theorem 4.4.10, we conclude that R_P, is F-regular for every $P \in U$.

■

Proposition 4.4.16. *If R is a F-regular Noetherian domain, then it is normal.*

Proof. Take $\dfrac{x}{y} \in \operatorname{Frac}(R)$ integral over R, we show that $\dfrac{x}{y} \in R$.

Since $\dfrac{x}{y}$ is integral over R, we have an integral equation

$$\left(\frac{x}{y}\right)^n + r_1 \left(\frac{x}{y}\right)^{n-1} + \cdots + r_n = 0$$

with $r_i \in R$, $\forall i = 1, \ldots, n$.

Note that we have the isomorphism

$$\begin{aligned} R[z] / \left\langle z^n + r_1 z^{n-1} + \cdots + r_n \right\rangle &\to R\left[\frac{x}{y}\right] \\ f + \left\langle z^n + r_1 z^{n-1} + \cdots + r_n \right\rangle &\mapsto f\left(\frac{x}{y}\right), \end{aligned}$$

with z an indeterminate.

Hence, $R\left[\frac{x}{y}\right]$ is a finite integral extension of R. Using the division algorithm, we obtain that the generators are

$$\left\{1, \frac{x}{y}, \left(\frac{x}{y}\right)^2, \ldots, \left(\frac{x}{y}\right)^{n-1}\right\}.$$

Thus, there exists $c \neq 0$ such that $cR\left[\frac{x}{y}\right] \subseteq R$.

Then $c\left(R\left[\frac{x}{y}\right]\right)^{p^e} \subseteq R$ for all $e \in \mathbb{N}$. This is, $cx^{p^e} \in \left(y^{p^e}\right)$, with $e \geq 0$. We have that

$$\begin{aligned} cx^{p^e} =& s_e y^{p^e} \text{for some } s_e \in S \\ \Rightarrow F_*^e\left(cx^{p^e}\right) =& F_*^e\left(s_e y^{p^e}\right) \\ \Rightarrow xF_*^e(c) =& yF_*^e(s_e) \end{aligned}$$

Finally, since R is F-regular, there exists $\pi \in \operatorname{Hom}(F_*^e R, R)$, such that $\pi(F_*^e c) = 1$. Applying π, we get

$$x = y\left(F_*^e(s_e)\right).$$

Thus $x \in (y)$, and so, y divides x. We conclude that $\frac{x}{y} \in R$. ■

Proposition 4.4.17. *If R be a F-regular Noetherian domain, then R is Cohen-Macaulay.*

Proof. We may assume, without loss of generality, that $(R, \mathfrak{m}, K)$ is a complete local domain. Let $x_1, \ldots, x_n$ be a system of parameters for R. We proceed by contradiction. Suppose these elements do not form a regular sequence. Then for some i there exists $z \notin (x_1, \ldots, x_{i-1})$ such that $zx_i \in (x_1, \ldots, x_{i-1})$. Thus, $z^{p^e}x_i^{p^e} \in \left(x_1^{p^e}, \ldots, x_{i-1}^{p^e}\right)$ for $e \geq 1$, and so, we get that

$$z^{p^e}x_i^{p^e} = r_1 x_1^{p^e} + \cdots + r_{i-1}x_{i-1}^{p^e}.$$

Take $A = K[\![x_1, ..., x_n]\!]$. Note that $A \subseteq R$ is a complete regular domain and it is a finitely generated R-module. Consider the inclusions

$$A \hookrightarrow A[z] \hookrightarrow R.$$

For each of these rings $\{x_1, \ldots, x_d\}$ is a system of parameters.

By Theorems 4.4.6 and Proposition 4.4.16, A is normal, so we take f a minimal polynomial over A for z. Observe that

$$A[z] \cong A[t] / (f(t)) \cong K[\![x_1, ..., x_n, t]\!] / (f).$$

This is $A[z]$ is a Cohen-Macaulay ring. Denote $B = A[z]$. By Theorem 2.0.26, there exists $\pi \in \operatorname{Hom}_B(R, B)$ such that $\pi(1) = c$, for some $c \neq 0$. Applying π, we get an equation in B:

$$\begin{aligned} \pi\left(z^{p^e} x_i^{p^e}\right) &= \pi\left(r_1 x_1^{p^e} + \cdots + r_{i-1} x_{i-1}^{p^e}\right) \\ z^{p^e} x_i^{p^e} \pi(1) &= \pi(r_1) x_1^{p^e} + \cdots + \phi(r_{i-1}) x_{i-1}^{p^e} \\ z^{p^e} x_i^{p^e} c &= \pi(r_1) x_1^{p^e} + \cdots + \pi(r_{i-1}) x_{i-1}^{p^e}. \end{aligned}$$

Now, since B is Cohen-Macaulay, $\left\{x_1^{p^e}, \ldots, x_n^{p^e}\right\}$ is a system of parameters. Then, $x_i^{p^e}$ is a non-zero divisor in $A[z] / \left(x_1^{p^e}, \ldots, x_{i-1}^{p^e}\right) A[z]$. Thus for every $e \in \mathbb{N}$,

$$cz^{p^e} \in \left(x_1^{p^e}, \ldots, x_{i-1}^{p^e}\right) A[z] \subseteq \left(x_1^{p^e}, \ldots, x_{i-1}^{p^e}\right) R,$$

which tells us that $cz^{p^e} = s_1 x_1^{p^e} + \dot{+} s_{i-1} x_{i-1}^{p^e}$ with $s_i \in R$. Hence

$$zF_*^e c = x_1 F_*^e s_1 + \cdots + x_{i-1} F_*^e s_{i-1}$$

Seeing that R is F-regular, there exists $\pi : F_*^e R \to R$ such that $\pi(F_*^e c) = 1$ for a large e. Applying π, we have that $z \in (x_1, \ldots, x_{i-1})$. Which is a contradiction. Thus, R is Cohen-Macaulay. ■

4.5 F-pure domians

The next F-singularity we aim to study is F-purity. This definition follows from the usual definition of purity for maps.

Definition 4.5.1. *Let R be a ring. An exact sequence of R-modules*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

*is called **pure** if for every R-module M*

$$0 \longrightarrow E' \otimes M \longrightarrow E \otimes M \longrightarrow E'' \otimes M \longrightarrow 0$$

is exact.

Definition 4.5.2. *Let R be a ring. A morphism of R-modules*

$$E' \longrightarrow E$$

*is called **pure** if for every R-module M*

$$E' \otimes M \longrightarrow E \otimes M$$

*is injective. In particular, if the Frobenius map is pure, we say that R is F**-pure**.*

An example of a F-pure ring is the following definition.

Definition 4.5.3 ([CMSV18]). *Let $w_1, \ldots, w_d$ be variables. For an integer j such that $1 \leq j \leq d$, we denote by W_j the $j \times (d+1-j)$ Hankel matrix, which has the following entries*

$$W_j = \begin{pmatrix} w_1 & w_2 & \cdots & w_{d+1-j} \\ w_2 & w_3 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ w_j & \cdots & \cdots & w_d \end{pmatrix}$$

For $1 \leq t \leq \min\{j, d+1-j\}$, the ideal $I_t(W_j)$ of $R = k[x_1, \ldots, x_d]$ is the one generated by the t-minors of W_j.

Any F-pure ring satifies the vanishing theorem in its sheaf cohomology [HH89b]. To understand this concept, we show some properties from pure maps.

Proposition 4.5.4. *Let R be a domain and $\varphi : M \to N$ an R-linear map that splits. Then, φ is pure.*

Proof. As φ splits, we have that $N = M \oplus S$ for some R-module S. Consider an R-module T. Then

$$N \otimes_R T = (M \otimes_R T) \oplus (S \otimes_R T).$$

Hence the map $\varphi \otimes_R Id_T : M \otimes_R T \to N \otimes_R T$ is injective. We conclude φ is pure. ∎

Proposition 4.5.5. *Let $\varphi : R \to A$ be a faithfully flat extension of domains. Then, φ is pure.*

Proof. Consider the following exact sequence

$$0 \longrightarrow \operatorname{Ker}(\varphi) \longrightarrow R \longrightarrow A \longrightarrow \operatorname{Coker}(\varphi) \longrightarrow 0.$$

It induces the exact sequence

$$0 \longrightarrow \operatorname{Ker}(\varphi) \otimes_R A \longrightarrow A \longrightarrow A \otimes_R A \longrightarrow \operatorname{Coker}(\varphi) \otimes_R A \longrightarrow 0.$$

Note that the map $A \to A \otimes_R A$ is injective, so $\operatorname{Ker}(\varphi) \otimes_R A = 0$. Now let T be a R-module. We have that the map $T \otimes_R A \to T \otimes_R A \otimes_R A$ is injective. As A is a faithfully flat R-module, this happens if and only if the map

$$R \otimes_R T \to A \otimes_R T$$

is injective. Hence, φ is pure. ∎

Definition 4.5.6. *Consider the homomorphism of free R-modules of finite rank*

$$\varphi : R^{\oplus n} \to R^{\oplus m}.$$

We define the map

$$\varphi^* : \operatorname{Hom}(R^{\oplus m}, R) \to \operatorname{Hom}(R^{\oplus n}, R).$$

If M is a free module of finite rank, then we define

$$M^* = \operatorname{Hom}(M, R).$$

We stick with this notation throughout the rest of this chapter.

Theorem 4.5.7. *Let R be a ring,*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

be an exact sequence, and

$$\varphi : F_1 \longrightarrow F_0$$

be a homomorphism of free modules of finite rank. Let $M = \operatorname{Coker} \varphi$ *and* $M' = \operatorname{Coker} \varphi^*$. *Then*

$$\operatorname{Ker}\left(M' \otimes E' \to M' \otimes E\right) \cong \operatorname{Coker}\left(\operatorname{Hom}(M, E) \to \operatorname{Hom}(M, E'')\right).$$

Proof. Consider

$$E : 0 \longrightarrow E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \longrightarrow 0$$

and

$$F : \qquad F_1 \xrightarrow{\varphi} F_0 \xrightarrow{\pi} M \longrightarrow 0.$$

We have the following sequences

$$0 \longrightarrow E''^* \xrightarrow{\beta^*} E^* \xrightarrow{\alpha^*} E'^*$$

and

$$0 \longrightarrow M^* \xrightarrow{\pi^*} F_0^* \xrightarrow{\varphi^*} F_1^* \xrightarrow{\theta} M' \longrightarrow 0.$$

Form the following double complex

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
M^* \otimes E' & \longrightarrow & M^* \otimes E & \longrightarrow & M^* \otimes E'' \\
\downarrow & & \downarrow & & \downarrow \\
F_0^* \otimes E' & \longrightarrow & F_0^* \otimes E & \longrightarrow & F_0^* \otimes E'' \\
\downarrow & & \downarrow & & \downarrow \\
F_1^* \otimes E' & \longrightarrow & F_1^* \otimes E & \longrightarrow & F_1^* \otimes E''.
\end{array}$$

The modules F_i are free, so they are projective. In addition, $\operatorname{Hom}(F_i, \bullet) \cong \operatorname{Hom}(F_i, R) \otimes \bullet$, so we get

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^* \otimes E' & \longrightarrow & M^* \otimes E & \longrightarrow & M^* \otimes E'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0^* \otimes E' & \longrightarrow & F_0^* \otimes E & \longrightarrow & F_0^* \otimes E'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1^* \otimes E' & \longrightarrow & F_1^* \otimes E & \longrightarrow & F_1^* \otimes E'' & \longrightarrow & 0.
\end{array}$$

By the right exactness of the tensor product, we have the sequence

$$F_0^* \otimes G \xrightarrow{\varphi^* \otimes \mathrm{Id}_G} F_1^* \otimes G \xrightarrow{\theta \otimes \mathrm{Id}_G} M' \otimes G \longrightarrow 0.$$

for every R-module G. Then, $M' \otimes G \cong \operatorname{Coker}(\varphi^* \otimes G)$. Now we complete the complex and simplify the notation

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^* \otimes E' & \longrightarrow & M^* \otimes E & \longrightarrow & M^* \otimes E'' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0^* \otimes E' & \longrightarrow & F_0^* \otimes E & \longrightarrow & F_0^* \otimes E'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1^* \otimes E' & \longrightarrow & F_1^* \otimes E & \longrightarrow & F_1^* \otimes E'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' \otimes E' & \longrightarrow & M' \otimes E & \longrightarrow & M' \otimes E'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}$$

Applying the Snake's Lemma, we have the exact sequence

$$M^* \otimes E' \longrightarrow M^* \otimes E \longrightarrow M^* \otimes E'' \xrightarrow{d} M' \otimes E' \longrightarrow M' \otimes E \longrightarrow M' \otimes E''.$$

We have that

$$\frac{M^* \otimes E''}{\mathrm{Ker}\,(d)} \cong \mathrm{Im}\,(d)$$

Note that, $\mathrm{Ker}\,(d) = \mathrm{Im}\,(\mathrm{Id}_{M^*} \otimes \beta)$ y $\mathrm{Im}\,(d) = \mathrm{Ker}\,(\mathrm{Id}_{M'} \otimes \alpha)$. Therefore,

$$\begin{aligned}\mathrm{Coker}\,(\mathrm{Id}_{M^*} \otimes \beta) &= \frac{M^* \otimes E''}{\mathrm{Im}\,(\mathrm{Id}_{M^*} \otimes \beta)} \\ &= \frac{M^* \otimes E''}{\mathrm{Ker}\,(d)} \\ &= \mathrm{Im}\,(d) \\ &= \mathrm{Ker}\,(\mathrm{Id}_{M'} \otimes \alpha)\,.\end{aligned}$$

■

Corollary 4.5.8. *Let R be a Noetherian ring. Then the exact sequence*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

is pure if and only if for every finitely presented module N, the morphism

$$\theta : \mathrm{Hom}\,(N, E) \to \mathrm{Hom}\,(N, E'')$$

is surjective.

Proof. For this part, we use a finitely presented module N. Then, we have an exact sequence

$$0 \longrightarrow K \xrightarrow{\;w\;} F \xrightarrow{\;\varphi\;} N \longrightarrow 0$$

where both K and F are free module of finite rank. Consider the R-module map

$$w^* : F^* \to K^*.$$

Note that F^* and K^* are finitely generated. Hence, $\mathrm{Im}\,(w^*)$ is finitely generated. Let $M = \mathrm{Coker}\,(w^*)$, which is finitely presented.

Now, suppose the exact sequence from the statement is pure. By Theorem 4.5.7, we have the exact sequence

$$\mathrm{Hom}\,(N, E) \xrightarrow{\;j\;} \mathrm{Hom}\,(N, E'') \xrightarrow{\;d\;} M \otimes E' \xrightarrow{\;h\;} M \otimes E.$$

Since h is injective, we get $\operatorname{Im} d = \operatorname{Ker} d = 0$. We conclude that j is surjective.

For the converse, we use that M is finitely presented. Note that $w^{**} = w$ and $\operatorname{Coker} w^{**} = N$. Likewise, applying Theorem 4.5.7, we get the exact sequence

$$\operatorname{Hom}(M, E) \longrightarrow \operatorname{Hom}(M, E'') \xrightarrow{\tilde{d}} N \otimes E' \xrightarrow{\tilde{h}} N \otimes E.$$

Then $\operatorname{Ker} \tilde{h} = \operatorname{Im} \tilde{d} = 0$. As the functor $\bullet \otimes N$ is right-exact, we conclude that the exact sequence in the statement is pure.

■

Corollary 4.5.9. *Let R be a Noetherian ring. Then the exact sequence*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0.$$

If E'' is finitely presented, then the exact sequence is pure if and only if it splits.

Proof. Let $\theta : E \to E''$ be the morphism in the statement. Suppose the sequence is pure. By the first part, we have

$$\operatorname{Hom}(E'', E) \longrightarrow \operatorname{Hom}(E'', E'') \longrightarrow 0.$$

In particular, there exists a $\varphi \in \operatorname{Hom}(E'', E)$ such that it is the preimage of the identity homomorphism. Therefore

$$(\varphi \circ \theta)(1) = 1.$$

Finally, let N be a finitely presented module, and be φ a splitting for θ. We show that for every $\beta \in \operatorname{Hom}(N, E'')$, there exists an element in $\alpha \in \operatorname{Hom}(N, E)$ such that $\theta \circ \alpha = \beta$. Take $\alpha = \varphi \circ \beta$. Then

$$\begin{aligned}(\theta \circ \alpha)(x) &= (\theta \circ \varphi \circ \beta)(x) \\ &= \beta(x).\end{aligned}$$

By Corollary 4.5.8, we are done. ■

Corollary 4.5.10. *Let $\varphi : M \to N$ be a pure map of R-modules. Then φ is split if $N/\varphi(M)$ is finitely presented.*

Proof. Follows from Corollary 4.5.9. ■

Theorem 4.5.11. *Let R be a Noetherian subring of S. Then*

$$\psi : R \longrightarrow S$$

is pure if and only if R is a direct summand of each finitely generated R-module N of S such that $R \subseteq N$. In fact, if S is module-finite over R, then ψ is pure if and only if R is a direct summand of S.

Proof. First suppose that ψ is pure. Since $R \otimes_R M \hookrightarrow S \otimes_R M$, we have that $R \otimes_R M \hookrightarrow N \otimes_R M$ is injective for every R-module M. By the Corollary 4.5.9, the sequence splits, this is $N \cong R \oplus \operatorname{Coker} i$.

Now,we show that ψ is pure. We may think S as an R-module, so it is finitely generated and thus it is finitely presented. Therefore, $S \cong R \oplus M$ for some R-module M, this is ψ splits. By Corollary 4.5.9, ψ is pure.

The second part follows. ∎

Corollary 4.5.12. *Let R be an excellent Noetherian domain whose fraction field is F-finite. Then R is F-split if and only if it is F-pure.*

Proof. By Theorem 4.2.5, R is F-finite. By Corollary 4.5.8, we are done. ∎

4.6 F-pure regular domains

In last section of this chapter we define F-pure regularity and its properties. For instance, it is cosely related to F-purity and similar F-splitting.

Definition 4.6.1. *Let c be an element in a domain R. Then R is said to be F**-pure along** c if there exists $e > 0$ such that the map*

$$\begin{aligned} \lambda_c^e : R &\to F_*^e R \\ 1 &\mapsto F_*^e c \end{aligned}$$

*is pure as R-module map. Moreover, if R is F-pure along c for every $c \in R - \{0\}$, then it is said to be F**-pure regular**.*

Remark 4.6.2. A domain R is F-pure if and only if it is F-pure along 1.

Remark 4.6.3. By Theorem 4.2.5 and Corollary 4.5.12. If R is an F-finite Noetherian domain, then the map λ_c^e is pure if and only if it splits.

Remark 4.6.4. If R is F-finite, then F-pure regularity is equivalent to F-regularity.

Lemma 4.6.5. *Let R be a domain an A and R-algebra.*

1. *If $M \to N$ and $N \to Q$ are pure homomorphisms of R-modules, then the composition is also pure.*

2. *If a composition of R-module maps*

$$M \to N \to Q$$

   is pure, then $M \to N$ is pure.

3. *If $M \to N$ is a pure R-linear map, then*

$$A \otimes_R M \to A \otimes_R N$$

   is a pure A-linear map.

4. *If $M \to N$ is a pure A-linear map, then it is also pure as R-linear map.*

5. *The R-linear map $M \to N$ is pure if and only if $M_P \to N_P$ is pure for each $P \in Spec(R)$;*

6. *A faithfully flat map is pure.*

7. *Let $(\Lambda, \leq)$ be a direct set with a least element λ_0, $\{N_\lambda\}_{\lambda \in \Lambda}$ be a direct limit system of R-modules, and $M \to N_{\lambda_0}$ be a R-linear map. Then $M \to \lim_{\to \lambda} N_\lambda$ is pure if and only if $M \to N_\lambda$ is pure $\forall \lambda \in \Lambda$.*

8. *Let $(R, \mathfrak{m})$ be a local ring. Then a map of modules $R \to N$ is pure if and only if $E \otimes_R R \to E \otimes_R N$ is injective, where E is the injective hull of $R/\mathfrak{m}$.*

Theorem 4.6.6. *Let R be a domain.*

1. *If R is F-pure along a product cd, then it is also F-pure along c and d. In paricular, if R is F-pure along some element, then R is F-pure.*

2. *Let R be F-pure regular. If $W \subseteq R$ is a multiplicative set, then R_W is F-pure regular.*

3. *Let $\varphi : R \to T$ be a pure homomorphism of domains. If T is F-pure regular, then R is also F-pure regular. In particular, if φ is faithfully flat and T is F-pure regular, then R is F-pure regular.*

4. *Let $R_1, \ldots, R_n$ be domains. If $R = R_1 \times \cdots \times R_n$ id F-pure regular, then each R_i is F-pure regular.*

Proof.

1. As the map $\times d$ is R-linear, so it is $\times F_* d$. On the other hand, R is F-pure along cd, so there exists $e > 0$ such that λ^e_{cd} is pure. Note that $\lambda^e_{cd} = \times F_* d \circ \lambda^e_c$. By Lemma 4.6.5, we get that λ^e_c is pure. Likewise, we get that λ^e_d is pure, because R is commutative. For the second part, suppose R is F-pure along an element c. As $c = c \cdot 1$, we have that R is F-pure along 1. By Remark 4.6.2, R is F-pure.

2. Let $a \in R_W \setminus \{0\}$. The $a = \dfrac{c}{d}$ with $c \neq 0$ and $d \in W$. As R is F-pure along c, there exists $e > 0$ such that the map λ^e_c is pure. By Lemma 4.6.5 and the isomorphism $(F^e_* R)_W \cong F^e_* R_W$, thet map $\lambda^e_{c/1}$ is also pure.

   On the other hand, the map
   $$\begin{aligned} \psi_{1/d} : R_W &\to R_W \\ 1 &\mapsto \frac{1}{d} \end{aligned}$$
   is an isomorphism of R_W-modules. Hence the R_W-module map
   $$\begin{aligned} F^e_* \psi_{1/d} : F^e_* R_W &\to F^e_* R_W \\ 1 &\mapsto \frac{1}{F^e_* d} \end{aligned}$$
   is also an isomorphism of R_W-modules. As $\psi_{1/d}$ is pure, we get that $F^e_* \psi_{1/d}$ is a pure R_W-linear map. Therefore, the composition $F^e_* \psi_{1/d} \circ \lambda^e_{c/1}$ is a pure R_W-linear map. Since $F^e_* \psi_{1/d} \circ \lambda^e_{c/1} = \lambda^e_{c/d}$, we are done.

3. First, note that φ is injective, because it is pure. Take $c \in R \setminus \{0\}$. Then T is F-pure along $\varphi(c)$. And so, there exists $e > 0$ such that $\lambda^e_{\varphi(c)}$ is a pure T-linear map.

Consider the composition $\varphi \circ \lambda^e_{\varphi(c)}$. It is also a pure T-linear map, and so, a pure R-linear map. In addition, we have the following commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & T \\ \Big\downarrow{\scriptstyle \lambda^e_c} & & \Big\downarrow{\scriptstyle \lambda^e_{\varphi(c)}} \\ F^e_* R & \xrightarrow{F^e_*(\varphi)} & F_* e T \end{array}$$

By Lemma 4.6.5, λ^e_c is pure. Now, if φ is faithfully flat, then it is pure by Lemma 4.6.5.

4. Note that the set

$$W = R_1 \times \cdots \times R_{i-1} \times \{1\} \times R_{i+1} \times \cdots \times R_n$$

is a multiplicative set of R. In addition, the map

$$\begin{aligned} \pi : R_S &\to R_i \\ \left(\frac{r_1}{s_1}, \ldots, \frac{r_i}{1}, \ldots, \frac{r_n}{s_n}\right) &\mapsto r_i \end{aligned}$$

is an isomorpshism. Thus, we show that R_W is F-pure regular. By the number 2 of this theorem, we are done.

■

Remark 4.6.7. Let R be F-pure along some element c. Then R is F-pure. Moreover, the map λ^e_c is pure. This implies that

$$\begin{aligned} F_*\lambda^e_c : F_*R &\to F_*\left(F^e_* R\right) \\ F_*1 &\mapsto F_*\left(F_*c\right) \end{aligned}$$

is a pure F_*R-linear map. By Lemma 4.6.5, it is also a R-linear map. This the composition $F_*\lambda^e_c \circ F$ is F-pure. Note that $F_*\lambda^e_c \circ F = \lambda^{e+1}_c$. We conclude that for each $n \geq e$, the map λ^{e+1}_c is pure.

Theorem 4.6.8. *A regular local ring $(R, \mathfrak{m})$ is F-pure regular.*

Proof. Note that $\cap_{e>0}\mathfrak{m}^{[p^e]} = 0$, by the Krull Intersection theorem. Let $c \in R \setminus \{0\}$. Thus there exists $e > 0$ such that $c \notin \mathfrak{m}^{[p^e]}$. We now prove that λ_c^e is pure. Let E be the injective hull of $R/\mathfrak{m}$. By Lemma 4.6.5, it suffices to check that $\lambda_c^e \otimes E$ is injective.

Let $\{x_1, \dots x_n\}$ be the minimal set of generators for $\mathfrak{m}$. As

$$E = \lim_{t\to \inf} R/\left(x_1^t, \dots, x_n^t\right),$$

we have that

$$F_*^e R \otimes E = \lim_{t\to \inf} R/\left(x_1^{tp^e}, \dots, x_n^{tp^e}\right).$$

The element $\lambda_c^e \otimes E\left(1 \otimes e\right) = F_*^e c \otimes e$. Therefore, as $c \notin \mathfrak{m}^{[p^e]}$, it is not zero in $F_*^e R \otimes E$. This is, the socle of E in $R/\mathfrak{m}$ is not in the kernel of $\lambda_c^e \otimes E$, hence it is an injective map. By Lemma 4.6.5, $\lambda_c^e \otimes E$ is pure. We conclude $(R, \mathfrak{m})$ is F-pure regular. ■

Proposition 4.6.9. *Let R be a F-pure regular domain. Then R is normal.*

Proof. Let $\dfrac{r}{s} \in \operatorname{Frac}(R)$ be an integral element over R. Then there exists a minimal polynomial $f(x) = x^n + a_1 x^{n-1} + \cdots + a_0$, with $a_i \in R$, such that $f\left(\dfrac{r}{s}\right) = 0$. We have that

$$r^n + s a_1 r^{n-1} + \cdots + s^n a_0 = 0$$

which implies that $r^n \in (s)$ and $r \in \overline{(s)}$. Thus there exists $h \in \mathbb{N}$ such that for each n

$$(r, s)^{n+h} = (s)^n (r, s)^h$$

Let $c = s^h$. Then $s^h r^n \in (s)^n (r, s)^h \subseteq (s)^n$. Take $n = p^e$. Consider the map λ_c^e which is pure and induce the injective map

$$\begin{aligned} R/(s) &\to R/(s) \otimes_R F_*^e R \cong F_*^e \left(R/(s)\right) \\ [r] &\to F_*^e \left[r^n c\right]. \end{aligned}$$

Note that $[r]$ is in the kernel of this map, because $r^n \in (s)$. This is, $r \in (s)$, and so, $s|r$. We conclude that $\dfrac{r}{s} \in R$.

■

Proposition 4.6.10. *Let R be a domain. The set*

$$I = \{c \in R \mid R \text{ is not } F\text{-pure along } c\}$$

is closed under multiplication by R and $R \setminus I$ is multiplicative closed. Moreover, if I is closed under addition, then it is a prime ideal.

Proof. First we show that I is closed under multiplication by elements in R. Let $c \in I$ and $r \in R$. Proceed by contradiction. Suppose $rc \notin I$. Then R is F-pure along rc, which implies that it is also F-pure along c.

Secondly, let $c, d \notin I$. Then there exist e and l intergers such that the maps λ_c^e and λ_d^l are pure. Consider the induced pure R-linear map

$$\begin{aligned} F_*^e\left(\lambda_d^l\right) : F_*^e R &\to F_*^e\left(F_*^l R\right) \\ F_*^e 1 &\mapsto F_*^e\left(F_*^l d\right). \end{aligned}$$

Note that $F_*^e\left(\lambda_d^l\right) \circ \lambda_c^e = \lambda_{c^{p^e}}^{e+l}$ and it is pure. Thus, $c^{p^e} d \notin I$. Suppose $cd \in I$. As $c^{p^e-1} \in R$, $c^{p^e} d \in I$, which is a contradiction. In addition, if $R \setminus I \neq \emptyset$, then $1 \in R \setminus I$. Otherwise, R would be F-pure regular.

Finally, if I is closed under addition, then I is a prime ideal. ∎

Chapter 5

Frobenius in Valuation Rings

In this chapter we study the Frobenius map on valuation domains. Moreover, we relate F-singularities with this kind of rings. The main goal is describe how Frobenius acts on this class of rings.

5.1 Flatness and purity of Frobenius in Valuation rings

Before starting the study of F-singularities in valuation rings we recall an important property about their modules.

Lemma 5.1.1. *A finitely generated torsion free module over a valuation ring is free. In particular, a torsion free module over a valuation ring is flat.*

Proof. Let V be a valuation domain and M be a finitely generated torsion free V-module. Let $\{m_1, ..., m_n\}$ be a minimal set of generators of M.

We proceed by contradiction. Suppose there exists a non-trivial relation among the generators. Consider $v_1, \dots, v_n \in V$ be such that

$$v_1 m_1 + \cdots + v_n m_n = 0.$$

The set of ideals of V is totally ordered so, without loss of generality, we assume that

$$(v_i) \subseteq (v_1),$$

$\forall i = 2, \dots, n$. This is, $v_i = a_i v_1$ $\forall i = 2, \dots, n$. Hence, we have that

$$v_1 (m_1 + a_2 m_2 + \cdots + a_n m_n) = 0.$$

Since M is torsion free, we have that

$$m_1 + a_2m_2 + \cdots + a_nm_n = 0,$$

and so, $m_1 = -(+a_2m_2 + \cdots + a_nm_n)$. The minimal generating set is $\{m_2, ..., m_n\}$, which is a contradiction. Thus, $v_i = 0$ for every $i = 1, \ldots, n$.

We conclude that, $\{m_1, ..., m_n\}$ is a free generating set for M, and thus, M is free.

Now, for the second part, consider N a torsion free module. Then N is the direct union of every finitely generated subomdule G. Since the direct union of flat modules is flat, we get that N is flat. ■

Proposition 5.1.2. *Let $(V, \mathfrak{m}, K)$ be a valuation domian of characteristic p. Then the Frobenius map is faithfully flat.*

Proof. Note that, because V is a domain, F_*V is a torsion free V-module. By Lemma 5.1.1, F_*V is flat, and thus, F is a flat homomorphism. Let M be a nonzero V-module. There exists an associated prime P of M. We have that

$$\frac{V}{P} \otimes_V F_*V \hookrightarrow M \otimes_V F_*V.$$

Note that $0 \neq \frac{V}{\mathfrak{m}F_*V} \subseteq \frac{V}{PF_*V}$. Since $\frac{V}{P \otimes_V F_*V} \cong \frac{V}{PF_*V}$, we have that $M \otimes_V F_*V \neq 0$. We conclude that F_*V is faithfully flat.

■

Corollary 5.1.3. *Every valuation ring $(V, \mathfrak{m}, K)$ of prime characteristic is F-pure.*

Proof. By Theorem 4.5.5, F is pure, because it is faithfully flat. ■

5.2 F-finite Valuation Rings

Proposition 5.2.1. *Let K be an F-finite field. A valuation ring V of K is F-finite if and only if F_*V is a free V-module.*

Proof. First, suppose that F_*V is a free V-module. Since $K \otimes_R F_*V \cong F_*K$ as K-vector spaces, we have that $\operatorname{rank}_K F_*K = \operatorname{rank}_V F_*V$. By hypothesis, K is F-finite, so $[F_K : K] < \infty$. Thus, $\operatorname{rank} F_*V < \infty$, this is, V is F-finite.

Now, let V be F-finite. Then F_*V is finitely generated. As F_*V is torsion free, by Lemma 5.1.1, it is free. ■

Corollary 5.2.2. *Let V be a F-finite valuation domain. Then, V is F-split.*

Proof. As V is F-pure, we have the following exact sequence

$$0 \longrightarrow V \longrightarrow F_*V \longrightarrow F_*V/V \longrightarrow 0.$$

Note that F_*V is a finite rank torsion-free free module. Thus, F_*V/V is finitely generated. In addition, V is a finitely generated module. By Corollary 4.5.10, V is F-split. ∎

Lemma 5.2.3. *Let R be a Noetherian domain such that* $\operatorname{Frac}(R)$ *is F-finite. Then R is F-finite if and only if there exists $\psi \in \operatorname{Hom}_{R^p}(R, R^p)$ such that $\psi(1) \neq 0$.*

Proof. First, we suppose R is F-finite. Since $\operatorname{Frac}(R)$ is F-finite, it is F-split. Let π be a K^p-linear splitting of Frobenius. This is, $\pi(1) = 1^p$. Let $\{f_1^p, \ldots, f_n^p\}$ a generating set of R as R^p-module, and $\dfrac{a_i^p}{b_i^p}$ their images under π. Denote the restriction of π to R as ϕ. Now, let $c^p = \prod_{i=1}^n b_i^p \in R^p$. We get the map $\psi = c^p\phi$, which is R^p-linear. Note that $\psi : R \to R^p$, and $\psi(1) = c^p \neq 0$.

For the converse, suppose there exists $\psi \in \operatorname{Hom}_{R^p}(R, R^p)$ such that $\psi(1) \neq 0$. Consider the map

$$\begin{aligned} \theta : R &\to R^{\vee\vee} \\ r &\mapsto e_r, \end{aligned}$$

where $R^{\vee\vee} = \operatorname{Hom}_{R^p}(\operatorname{Hom}_{R^p}(R, R^p), R^p)$ and e_r is the evaluation map at r. Note that if $x \in R$ is a nonzero element, then we have the map $\gamma = \psi \circ (\times x^{p-1})$, and so $\gamma(x) = x^p\psi(1) \neq 0$. We see that θ is injective, take $x = z - y$ for any two different elements y, z in R.

We prove that $R^\vee = \operatorname{Hom}_{R^p}(R, R^p)$ is a R^p-finitely generated module. Indeed, let M be a maximal free R^p-submodule contained in R. Then

$$\operatorname{rank}(M) = \dim_{K^p}(M \otimes_{R^p} K^p) = \dim_{K^p} K = [K : K^p],$$

and so $M \otimes_{R^p} K^p \cong R \otimes_{R^p} K^p$ as K^p-vector spaces. Therefore $R/M \otimes_{R^p} K^p = 0$, which implies thath R/M is a torsion module. Considering the exact sequence

$$0 \longrightarrow M \longrightarrow R \longrightarrow R/M \longrightarrow 0,$$

we get that $R^\vee \hookrightarrow M^\vee$, because the dual of a torsion module is zero. Since M is finitely generated, we have that $M^\vee$ is also finitely generated as R^p-module. Thus, $R^\vee$ is finitely generated, and so $R^{\vee\vee}$ is a finitely generated R^p-module. Since $R \subseteq R^{\vee\vee}$, we conclude R is a finitely generated R^p-module. ■

Theorem 5.2.4. *Let R be a Noetherian domain whose fraction field is F-finite. If R is F-split, then it is F-finite.*

Proof. Let $\pi \in \operatorname{Hom}_{R^p}(R, R^p)$ be a splitting. Thus, $\pi(1) = 1$. By Lemma 5.2.3, R is F-finite. ■

Now we state the equivalence between F-splitness and F-finiteness, along with excellence. This is, we now state the main result of this thesis.

Corollary 5.2.5. *Let V be a discrete valuation domain whose field of fractions is F-finite. Then the following are equivalent:*

1. *V is F-split;*
2. *V is F-finite;*
3. *V is excellent.*

Proof. As V is a discrete valuation domain, it is Noetherian. Applying Theorems 5.2.4 and 5.2.2 we get the equivalence between F-splitness and F-finiteness. Now, Theorem 4.2.5 gives us the equivalence between F-finiteness and excellent rings. ■

5.3 F-pure regular Valuation Rings

Finally, we give some statements before the extended version of the main theorem.

Proposition 5.3.1. *Let $(V, \mathfrak{m})$ be a valuation domain. The set of elements along which V fails to be pure is the prime ideal*

$$\Phi = \cap_{e>0} \mathfrak{m}^{[p^e]}.$$

Proof. Let $F_*c \in \Phi$. Then $F_*c \in \mathfrak{m}^{[p^e]}$ for every $e > 0$. Consider the residue field of V, K. Then the map

$$\begin{aligned} \lambda_c^e \otimes_V K : V \otimes_V K &\to F_*^e V \otimes_V K \\ 1 \otimes_V k &\mapsto F_*^e c \otimes_V k. \end{aligned}$$

As $c^{p^e} \in \mathfrak{m}^{[p^e]}$, we have that $F_*^e c \otimes_V k = F_*^e 1 \otimes_V [0]$. Hence, λ_c^e is not pure for every $e > 0$.

Now, consider an element $c \notin \mathfrak{m}^{[p^e]}$ for some $e > 0$, and the set Σ of submodules of $F_*^e V$ containing $F_*^e c$. Then Σ is a directed set under inclusion with least element the module $(F_*^e c)$. Take $F_*^e N \in \Sigma$. There exists a map

$$\begin{aligned} \lambda_c^N : V &\to F_*^e N \\ 1 &\mapsto F_*^e c. \end{aligned}$$

In addition, by Lemma 5.1.1, $F_*^e N$ is free. Moreover, $F_*^e c \notin \mathfrak{m} F_*^e N$; otherwise, $F_*^e c \in F_*^e \mathfrak{m}^{[p^e]}$, and $c \in \mathfrak{m}^{[p^e]}$, which is a contradiction. By Nakayama's Lemma, $F_*^e c$ is part of a basis for $F_*^e N$. This implies that λ_c^N splits, and so it is pure.

On the other hand, if $F_*^e N \subset F_*^e M$ are elements of Σ, then we have the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\lambda_c^N} & F_*^e N \\ & \searrow^{\lambda_c^M} & \downarrow \\ & & F_*^e M. \end{array}$$

Therefore, we have a direct system consisting of elements

$$A_{F_*^e N} = F_*^e N$$

indexed by Σ, and injections as the morphisms from the definition. By Theorem 2.0.27, the direct limit is $F_*^e V$. As the map $\lambda_c^{(F_*^c)}$ is V-linear, $\lambda_c^e : V \to F_*^e V$ is pure, by Lemma 4.6.5.

Finally, by Proposition 4.6.10, Φ is a prime ideal. ■

Corollary 5.3.2. *For a valuation ring $(V, \mathfrak{m})$, the quotient V/Φ is a F-pure regular domain. Furthermore, V is F-pure regular if and only if $\Phi = 0$.*

Proof. Note that V/Φ is a domain whose ideals inherit the order from V. In addition, its unique maximal ideal is $\mathfrak{m} + \Phi$.

Let $x = \dfrac{r+\Phi}{s+\Phi} \in \operatorname{Frac}(V/\Phi)$ be such that $x \notin V/\Phi$. Then r does not divide s in V, and so, $\dfrac{r}{s} \notin V$. As V is a valuaton domain, $\dfrac{s}{r} \in V$. Hence $x^{-1} \in V/\Phi$.

Finally, since $\left[\bigcap_{e>0} \mathfrak{m}^{[p^e]}\right] = [0]$, we conclude that V/Φ is F-pure regular. The second statement follows from Proposition 5.3.1. ■

Theorem 5.3.3. *Let $(V, \mathfrak{m})$ be a valuation domain. Then V is F-pure regular if and only if it is either a field or a DVR.*

Proof. First, suppose that V is F-pure regular. We proceed by contradiction. Suppose there exists $P \in \operatorname{Spec}(V)$, such that $0 \subsetneq P \subsetneq \mathfrak{m}$.

Take $x \in \mathfrak{m} \setminus P$ and $c \in P \setminus \{0\}$. If $c | x^n$ for some n, then there exists $q \in V$ such that $x^n = qc$. This implies that $x^n \in (c) \subset P$. Thus $x \in P$, which is a contradiction. Hence $x^n | c$ for every n. In particular, take $n = p^e$ with $e \in \mathbb{N}$. We have that $c \in \left(x^{[p^e]}\right) \subset \mathfrak{m}^{[p^e]}$. This is, $c \in \Phi$. By Corollary 5.3.2, V is not F-pure regular, which is a contradiction. Therefore, V has dimension at most one.

Now, suppose that V is F-regular. We show Γ_V is isomorphic to $\mathbb{Z}$. Let h be the infimum of Γ_V. Then, h is positive. Indeed, let $c \in \mathfrak{m}$. Note that the sequence $\left\{\dfrac{v(c)}{p^e}\right\}_{e>0}$ converges to 0 when $e \to \infty$. In fact, suppose that there exists an element $x \in V$ such that for some e

$$0 < v(x) < \frac{v(c)}{p^e}.$$

Then $0 < v\left(x^{p^e}\right) < v(c)$, and so, $v\left(\dfrac{x^{p^e}}{c}\right) < 0$, this implies that $x^{p^e} | c$. Therefore, $c \in (x)^{[p^e]} \subset \mathfrak{m}^{[p^e]}$, which contradicts the fact that R is F-pure regular.

We show that $h \in \Gamma_V$. Note that there exists g such that $0 < g < h$. Suppose $h \notin \Gamma_V$. Then there exist elements $z, y \in \mathfrak{m}$ such that

$$h < v(z) < v(y) < h + g.$$

We have that $0 < v\left(\dfrac{y}{z}\right) < g < h$, which is a contradiction. Hence, $h \in \Gamma_V$.

Suppose there exists an element $\alpha \in \Gamma_V$ such that $h \nmid \alpha$. By the division algorithm $\alpha = qh + r$, with $q, r \in \mathbb{R}$. Therefore, $r \in \Gamma_V$ and $r < h$ which is a contradiction. Thus $\Gamma_V = \langle h \rangle$. This is, $\Gamma_V \cong \mathbb{Z}$, this isomorphism also preserves the order. Hence, by Theorem 3.3.6 V is Noetherian and by Corollary V 3.3.10 is a discrete valuation domain. ■

The following is the extended version of the Theorem 5.2.5. It gives us an extra equivalence which uses the strong relation between F-regularity and F-pure regularity.

Theorem 5.3.4 ([DS16]). *Let $(V, \mathfrak{m})$ be a discrete valuation domain with F-finite fraction field. The following are equivalent*

1. *V is F-split;*
2. *V is F-finite;*
3. *V is excellent;*
4. *V is F-regular.*

Proof. By Theorem 5.2.5, we know the equivalence among 1, 2 and 3.

Now, we will show that V is F-regular if and only if it is F-finite. Suppose V is F-regular. By Theorem 4.4.5, V is F-split. Therefore it is F-finite. Conversely, by Theorem 5.1.2, F_*V is faithfully flat in particular, it is flat. Hence by Kunz Theorem 4.1.14, V is regular. Finally, by Theorem 4.4.6, we have that V is F-regular. ■

Remark 5.3.5. We have that the ring of formal series $K[\![x]\!]$ satisfies Theorem 5.2.5 if and only if K is F-finite. Moreover, $K[\![x]\!]$ is F-finite if and only if K is F-finite. In addition, $K[\![x]\!]$ is always F-split and F-regular. Despite this, it is not a counter-example for Theorem 5.2.5, since this theorem requieres $\operatorname{Frac}(K[\![x]\!])$ to be F-finite.

However, not every discrete valuation domain with F-finite fraction field is F-finite. In order to give an example of this, we first mention a lemma.

Lemma 5.3.6 ([DS16]). *Let V be a valuation domain of an F-finite field K of prime characteristic p. If V is F-finite, then*

$$[\Gamma : p\Gamma]\,[k : k^p] = [K : K^p]$$

where Γ is the valuation group and k the residue field of V.

Example 5.3.7. Consider $\mathbb{F}_p(\!(t)\!)$ which is F-finite and the fraction field of $\mathbb{F}_p[\![t]\!]$. Note that we have a valuation

$$\begin{aligned}\tilde{v} : \mathbb{F}_p[\![t]\!] &\to \mathbb{Z}\\ t &\mapsto 1.\end{aligned}$$

which is discrete.

As $\mathbb{F}_p\left(t\right)$ is countable and $\mathbb{F}_p(\!(t)\!)$ is uncountable, there exists a trascendent element in $\mathbb{F}_p[\![t]\!]$ transcendental over $\mathbb{F}_p\left(t\right)$, namely,

$$f\left(t\right) = \sum_{n=1}^{\infty} a_n t^n.$$

The elements t and f are algebraically independent, so we have the injective map

$$\begin{aligned}\psi : \mathbb{F}_p\left[x, y\right] &\hookrightarrow \mathbb{F}_p[\![t]\!]\\ x &\mapsto t\\ y &\mapsto f.\end{aligned}$$

This induces the extension of fields

$$\mathbb{F}_p\left(x, y\right) \hookrightarrow \mathbb{F}_p(\!(t)\!).$$

We restrict the valuation $\tilde{v}$ to $\mathbb{F}_p\left(x, y\right)$, call it v, which is also discrete. Let V be the valuation domain associated to v. We have that $[L : L^p] = p^2$ and $[\Gamma_v : \Gamma_{v^p}] = [\Gamma_v : p\Gamma_v] = p$. On the other hand, let $u \in \mathbb{F}_p\left(x, y\right)$ with image in $\mathbb{F}_p(\!(t)\!)$

$$\sum_{n=0}^{\infty} b_n t^n.$$

If $u \in V$, then $v\left(u\right) \geq 0$. Therefore, $v\left(u - b_0\right) > 0$. This is, $u \sim b_0$ in $V/\mathfrak{m}_V$. Thus, we have

$$\begin{aligned}\left[k\left(v\right) : k\left(v^p\right)\right] &= \left[k\left(v\right) : k\left(v\right)^p\right]\\ &= \left[\mathbb{F}_p : \mathbb{F}_p^p\right]\\ &= \left[\mathbb{F}_p : \mathbb{F}_p\right]\\ &= 1,\end{aligned}$$

where $k\left(v\right)$ is the residue field of V. By Theorem 5.3.6, V is not F-finite.

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